Econometric Analysis via Filtering for Financial Ultra-High Frequency (UHF) Data

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UHF Data and Marked Point Process

Two Characteristics of UHF Data

- Observations occur at varying random time intervals
- Market microstructure noises (frictions) exist in price data

- **Marked Point Process:** $\{(T_n, X_n)\}$, where $\{T_n\}$ increasing, i.e. $T_n \leq T_{n+1}$.
- **Marked Poisson Process:** $\{(T_n, X_n)\}$ with $\{T_n\}$ from a conditional Poisson Process (or Cox process, or doubly stochastic Poisson).

An Irregularly-Spaced Time Series

- Engle (2000) – Data: $\{(\Delta t_i, y_i), i = 1, ..., N\}$ where $\Delta t_i = t_i - t_{i-1}$.
- **Conditional density:**

$$(\Delta t_i, y_i)|\mathcal{F}_{i-1} \sim f(\Delta t_i, y_i|\breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta)$$

where $\breve{z}_i = \{z_i, z_{i-1}, ..., z_1\}$.

$$f(\Delta t_i, y_i|\breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta) = g(\Delta t_i|\breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta) q(y_i|\Delta t_i, \breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta) \quad (1)$$

Similar structure as in Eq.(4)

- **Log likelihood:**

$$l(\Delta, Y; \theta) = \sum_{i=1}^{N} \log g(\Delta t_i|\breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta) + \sum_{i=1}^{N} \log q(y_i|\Delta t_i, \breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta)$$

Related Developments

- **Autoregressive Conditional Duration (ACD)** model by Engle and Russell (1998), and logarithmic ACD by Bauwens and Giot (2000), threshold ACD by Zhang, Russell and Tsay (2001), Asymmetric ACD by Bauwens and Giot (2003)
- Ordered Probit model by Hausman, Lo and Mackinlay (1992), Price Change Duration model by McCulloch and Tsay (2001), Activity -Direction -Size model by Rydberg and Shepherd (2003), Autoregressive Conditional Multinomial- ACD by Russell and Engle (2005)
- Bivariate Point process model by Engle and Lunde (2003), and Autoregressive Conditional Intensity model by Russell (1999)
- UHF-GARCH by Engle (2000), ACD-GARCH by Ghysels and Jasiak (1998), Engle and Zheng (2007)

- **The Second View:**
A Realized Sample Path of MPP: Zeng (2003)

Construction of Price from Value

- *Intrinsic value process:* $X(t)$

Assumption 1.1: Markov process, (θ, X), is the solution of a martingale problem for a generator $\mathbf{A}$ such that

$$M_f(t) = f(\theta(t), X(t)) - \int_0^t \mathbf{A} f(\theta(s), X(s)) ds$$

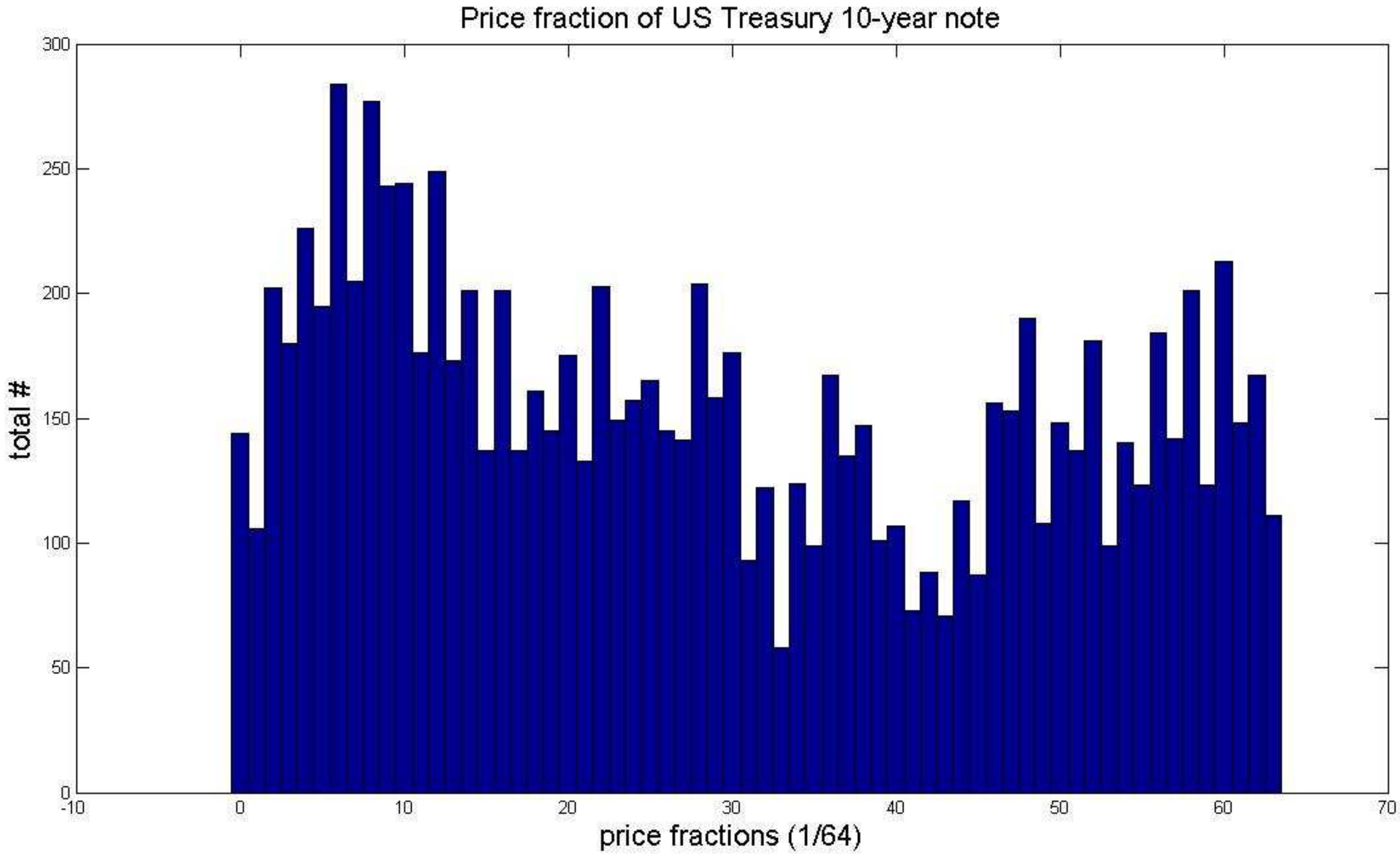
is a $\mathcal{F}_t^{\theta,X}$-martingale.

- *Trading times:* $t_1, t_2, \ldots, t_i, \ldots$ follows a *conditional Poisson process* with $a(\theta(t), X(t), t)$.
- *Price* at t_i: $Z(t_i) = F(X(t_i))$

where F, a random transformation modeling market microstructure noise, is specified by the transition probability $p(Z(t_i)|X(t_i); \theta)$.

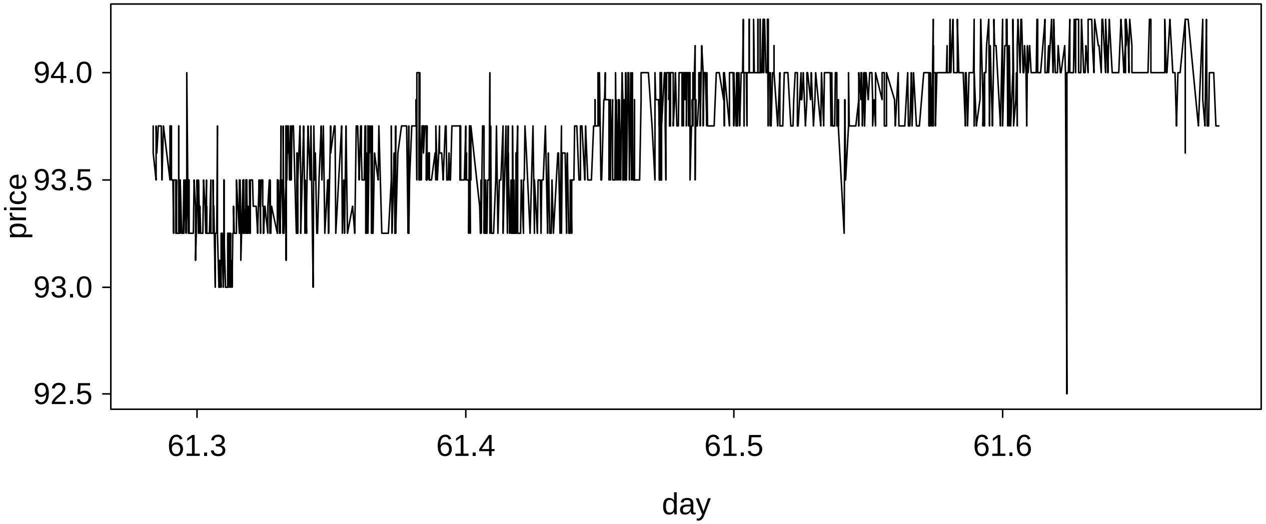
Remark: Related to (1) the time series structure models surveyed in Hasbrouck (1996) and (2) the Two-Scaled Realized Volatility (TSRV) Models for in Zhang, Mykland, and Aït-Sahalia (2005), Li and Mykland (2007), BHLS (2008) and many others.

Price Clustering in Treasury Notes

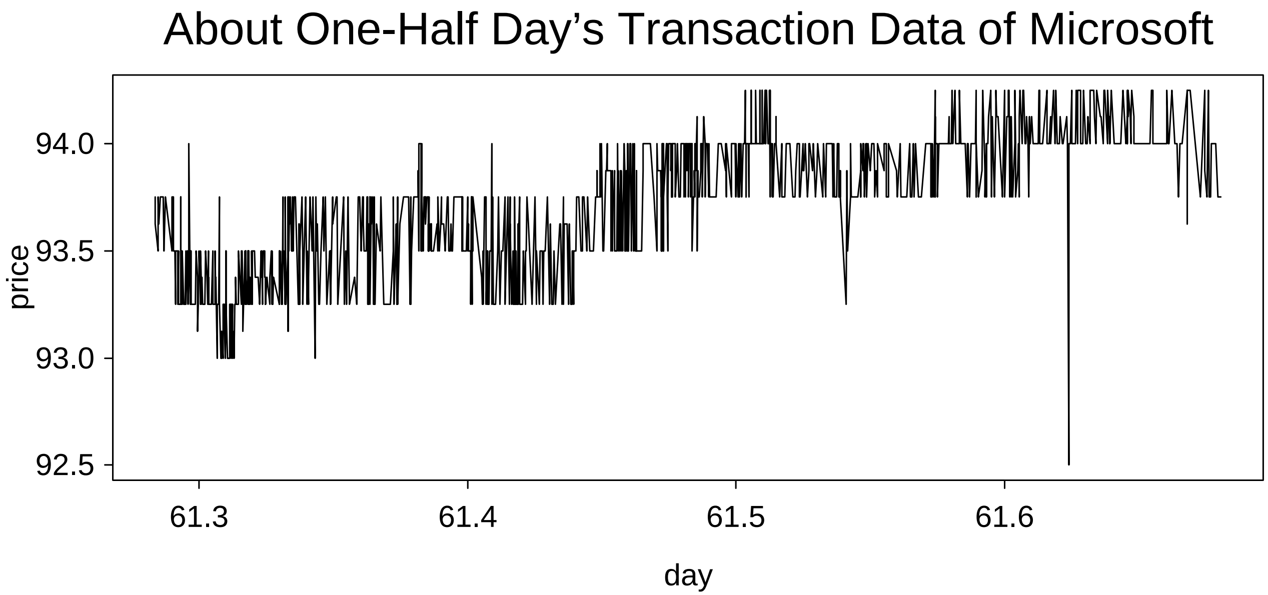


A Collection of Counting Processes

About One-Half Day's Transaction Data of Microsoft



A Collection of Counting Processes



$$\vec{Y}(t) = \begin{pmatrix} N_1(\int_0^t \lambda_1(\theta(s), X(s), s)ds) \\ N_2(\int_0^t \lambda_2(\theta(s), X(s), s)ds) \\ \vdots \\ N_n(\int_0^t \lambda_n(\theta(s), X(s), s)ds) \end{pmatrix}, \tag{3}$$

where $Y_j(t) = N_j(\int_0^t \lambda_j(\theta(s), X(s), s)ds)$ records the cumulative # of trades that have occurred at the jth price level up to time t.

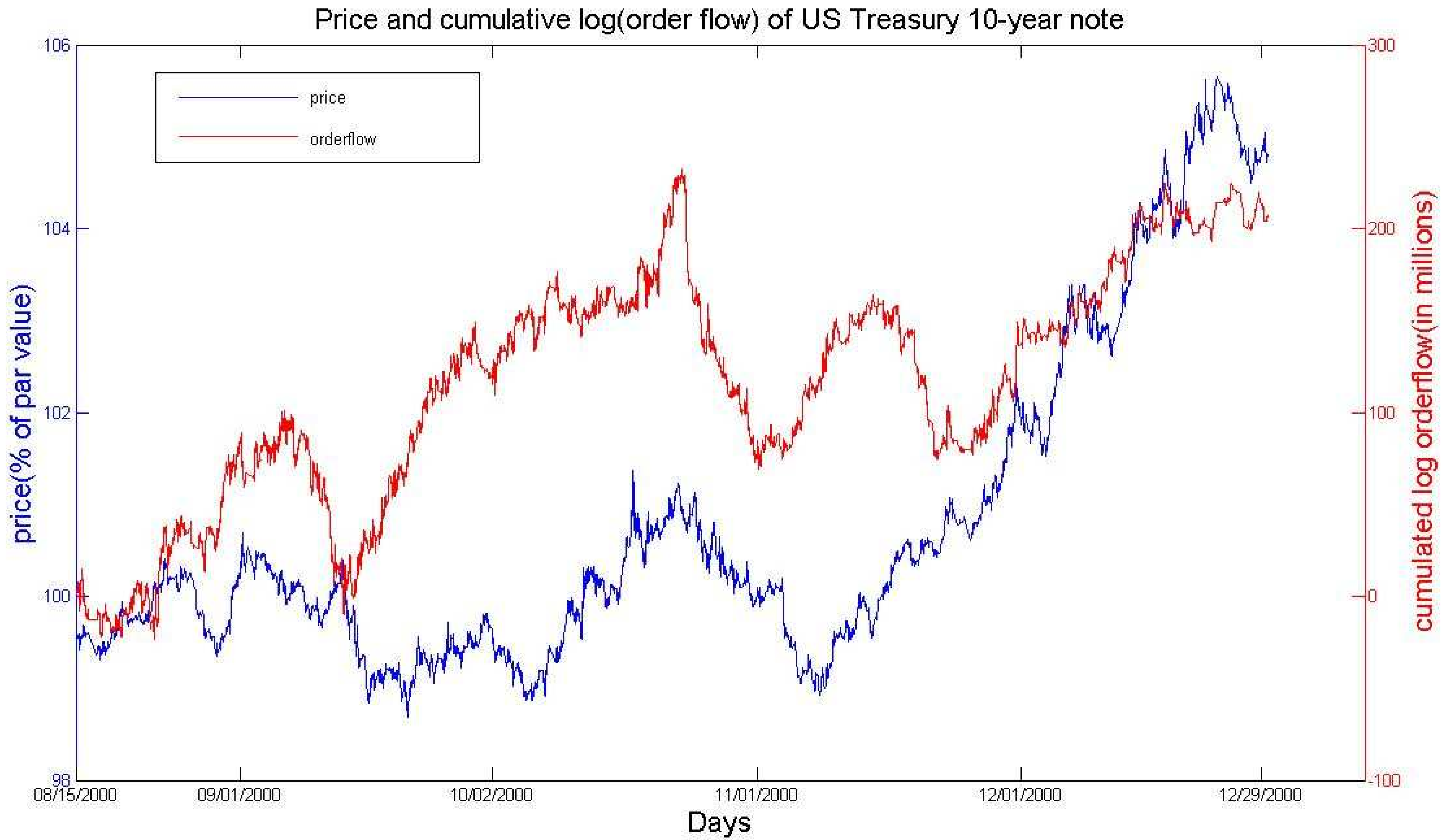
Other Assumptions of Model I

Filtering with counting process observations

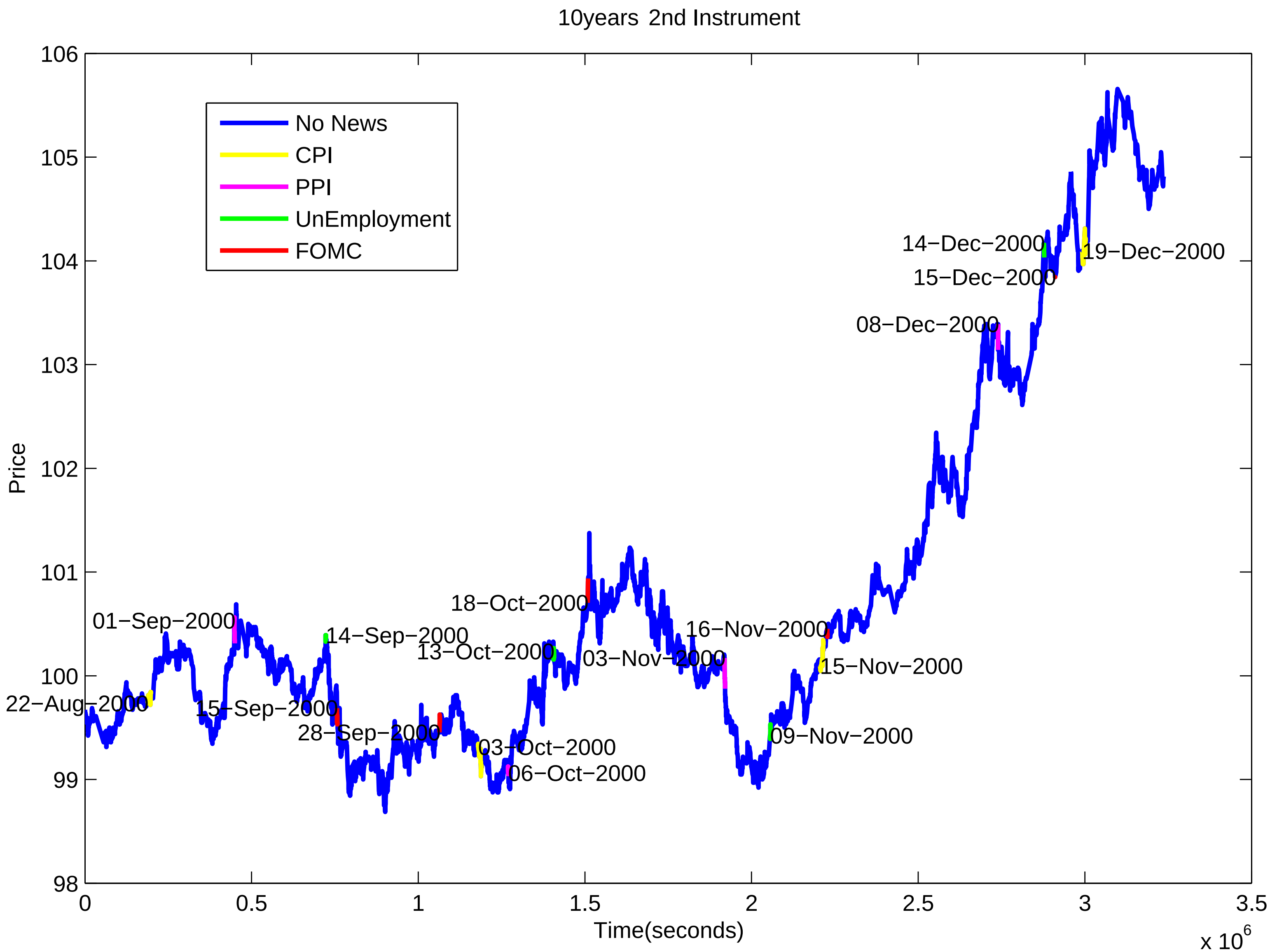
- **Assumption 1.2:** $(N_1, \ldots, N_n)$ are unit Poisson processes under measure P.
- **Assumption 1.3:** $(\theta, X), N_1, \ldots, N_n$ are independent under P.
- **Assumption 1.4:** $0 \le a(\theta(t), X(t), t) \le C$ for some $C > 0$ and all $\theta(t), X(t), t > 0$.
- **Assumption 1.5:** Intensities: $\lambda_j(\theta, x, t) = a(\theta, x, t)p(y_j|x)$, where $a(x, \theta, t)$ is the total trading intensity, and $p_j = p(y_j|x)$ is the transition probability from x to y_j.

Signal: (θ, X) **Observation:** $\vec{Y}$ or Z.

Price and Order Flow



Price and Economic Announcements



A Motivating Example

V_t: observable variable such as *signed order flow, economic news, a trading indicator or others* in Treasury market.

- **Intrinsic value process:**

$$\frac{dX_t}{X_t} = \mu dt + (\sigma + \kappa_1 V_1(t) + \kappa_2 V_2(t)) dB_t$$

- **Trading intensity:** $a(V^{t-}, Z^{t-}, t)$ can depend on the past history of V and Z, where $V^t(\cdot) = V(\cdot \wedge t)$. E.g. $a(V(t_{i-1}), Z(t_{i-1}), \Delta_{i-1}, t)$ for $t_{i-1} \leq t < t_i$. This allows *ACD Model*.
- **Price** at time t_i:

$$Z(t_i) = F(X(t_i))$$

where F is a random transformation specified by a transition probability $p(Z(t_i)|X(t_i); \theta)$.

Random-arrival-time State-Space Model

- **State process:** $X(t)$.

Assumption 2.1: is the same as Assumption 1.1, but both θ and X can be vector processes,

$$M_f(t) = f(\theta(t), X(t)) - \int_0^t \mathbf{A}_v f(\theta(s), X(s))ds$$

is a $\mathcal{F}_t^{\theta, X, V}$-martingale.

- **Event times:** $t_1, t_2, \ldots, t_i, \ldots$ follows a point process with a stochastic intensity $\bar{\lambda}(\theta(t), X(t), V^t, Z^t, t)$ where $V^t(\cdot) = V(\cdot \wedge t)$.
- **Observation** at t_i:

$$Z(t_i) = F(X(t_i))$$

where F is a random transformation with a transition probability $p(Z(t_i)|X(t_i); \theta(t_i), V^{t_i-}, Z^{t_i-}, t))$.

Filtering with MPP Observations

- **Setup:** Mark space: U; measure space: $(U, \mathcal{U}, \mu)$, μ: finite measure; ξ is a Poisson Random Measure (PRM) on $\mathcal{U} \times \mathcal{B}[0,\infty) \times \mathcal{B}[0,\infty)$ with mean measure $\mu \times m \times m$. For $A \in \mathcal{U}$,

$$Y(A,t) = \int_{A\times[0,t]\times[0,+\infty)} \mathbf{I}_{[0,\lambda(\theta(s-),X(s-),V^{s-},Z^{s-};u,s-)]}(v)\xi(du \times ds \times dv),$$

where $Y(A,t)$ is a counting process recording the cumulative number of events that have occurred in the set A up to time t.

$$\tilde{Y}(A,t) = Y(A,t) - \int_{A\times[0,t]} \lambda(\theta(s), X(s), V^s, Z^s; u, s)\mu(du)ds$$

is a martingale.

Signal: (θ, X) **Observation:** (Y, V) or (Z, V).

The Reference Measure

- Under the reference measure $\mathbf{Q}$, For $A \in \mathcal{U}$,

$$Y(A,t) = \int_{A\times[0,t]\times[0,+\infty)} \mathbf{I}_{[0,1]}(v)\xi(du \times ds \times dv),$$

where $Y(A,t)$ is a counting process recording the cumulative number of events that have occurred in the set A up to time t.

$$\tilde{Y}(A,t) = Y(A,t) - \int_{A\times[0,t]} \mu(du)ds = Y(A,t) - \mu(A)t$$

is a martingale.

- **For Model I:** $U = \{1, 2, \cdots, n\}$, $A = j$ and V and Z are not in the intensity,
Under $\mathbf{P}$, $Y(A,t) = Y(j,t) = N_j(\int_0^t \lambda_j(\theta(s), X(s), s)ds)$.
Under $\mathbf{Q}$, $Y(A,t) = Y(j,t) = Y_j(t)$ is a unit Poisson process.

Other Assumptions of Model II

- **Assumption 2.2:** ξ is a PRM with the mean measure $\mu \times m \times m$ under P, where μ is finite measure.
- **Assumption 2.3:** (θ, X) and ξ are independent under measure P.
- **Assumption 2.4:** For $t \in (t_i, t_{i+1}]$, there exists a positive constant, C, which may depend on $\{t_1, t_2, \cdots, t_i\}$ and t, such that $0 \leq \bar{\lambda}(\theta^t, X^t, V^t, Z^t, t) \leq C$.
- **Assumption 2.5:** Stochastic intensity kernel:

$$\lambda(u,t) := \lambda(u, t; \theta(t-), X(t-), V^{t-}, Z^{t-},)$$

$$= \bar{\lambda}(\theta(t-), X(t-), V^{t-}, Z^{t-}, t-) p(u|X(t); \theta(t-), , V^{t-}, Z^{t-}, t-) \qquad (4)$$

where $p(u|X(t); \theta(t-), V^{t-}, Z^{t-}, t))$ is the transition probability from $X(t)$ to u.

Remark: Note the similarity between Eq.(4) and Eq.(1).
(5)

Examples I

Group I: without continuous-time latent X_t

- $U = \{1\}$: Cox process and Exponential ACD model (Engle and Russell 1998).
- $U = \{\frac{a}{M}, \frac{a+1}{M}, \cdots, \frac{b}{M}\}$, or R, or R^+: UHF-GARCH model and many previously reviewed models under the framework of Engle (2000).

Group II: with latent X_t not depending on V

- Zeng (2003) and its extension to multi-stocks (Scott and Zeng 2008).
- Estimating Volatility via filtering: Frey and Runggaldier (2001), Cvitanic, Liptser and Rozovskii (2006), Ceci and Gerardi (2007a,b).
- Estimating Markov process sampled at conditional Poisson time: Duffie and Glenn (2004).
- Classical examples: Segall, Davis and Kailath (1975), Bremaud (1981), Liptser and Shiryayev (1978), Kliemann, Koch and Marchetti (1990), and Last and Brandt (1995).

Examples II

Group III: with latent X_t depending on V

- V_t can change at random times (such as trading time) or deterministic times (such as every 5 minutes or every day).
- V_t can takes values such as ± 1 for the indicator of buyer or seller initiating trade; or 0 or 1 for the indicator of a special period; or other values such as order flow or a function of order flow.
- When X_t is GBM, V_t can be added in instantaneous expected return, or instantaneous volatility, or in the trading noise.
- X_t can be O-U process; CIR model; CES model; SV models; plus jumps; plus regime-switching; with spikes; α-stable process; and others. Then, V_t can be added in the related parameters with different economic interpretations.

- Z can be trading prices or ask and bid quotes.

An Integral Form of Price

- Let $Z(t)$ be the price of the most recent transaction at or before time t.

$$Z(t) = Z(0) + \int_{[0,t]\times U} (u - Z(s-))Y(ds, du).$$

Remarks :

- This is the telescoping sum: $Z(t) = Z(0) + \sum_{t_i \le t}(Z(t_i) - Z(t_{i-1}))$. This form is similar to that of Skorohod (1989) for continuous time Markov chain.

- This form is essential for the *risk minimization hedging and option pricing* (Lee and Zeng 2009, for Model I), and the *mean-variance portfolio selection* problem of the model (Xiong and Zeng 2009, for Model I).

Joint Likelihood Function

- Continuous-time joint likelihood function of (θ, X, Y):
- For Model I,

$$L(t) = \frac{dP}{dQ}(t) = \exp\left\{ \sum_{k=1}^{n} \int_0^t \log \lambda_k(\theta(s-), X(s-), s-) dY_k(s) \right.$$
$$\left. - \sum_{k=1}^{n} \int_0^t \Big[\lambda_k(\theta(s-), X(s-), s) - 1 \Big] ds \right\}.$$

- For Model II,

$$L(t) = \exp\left\{ \int_0^t \int_U \log \lambda(u, s; \theta(s-), X(s-), V^{s-}, Z^{s-}) Y(ds, du) \right.$$
$$\left. - \int_0^t \int_U \Big[\lambda(u, s; \theta(s-), X(s-), V^{s-}, Z^{s-}) - 1 \Big] \mu(du) ds \right\}$$

Likelihoods and Posterior

Define: $\phi(f,t) = E^Q[f(\theta(t), X(t))L(t)|\mathcal{F}_t^{Y,V}]$. Then, $\phi(1,t) = E^Q[L(t)|\mathcal{F}_t^{Y,V}]$ is the *likelihood* of Y *or* the *integrated (marginal) likelihood* of Y after assigning a prior to $(\theta(0), X(0))$.

Define: π_t is the conditional distribution of $(\theta(t), X(t))$ given $\mathcal{F}_t^{Y,V}$.
π_t becomes the *posterior* after a prior is assigned.
Define: $\pi(f,t) = E^P[f(\theta(t), X(t))|\mathcal{F}_t^{Y,V}] = \int f(\theta, x)\pi_t(d\theta, dx)$.

- Bayes theorem gives:

$$\pi(f,t) = \frac{\phi(f,t)}{\phi(1,t)}.$$

SPDE for ϕ_t – unnormalized filtering equation
SPDE for π_t – normalized filtering equation

Bayes Factor and Likelihood Ratio

Suppose there are two models: Model 1 and Model 2.

Define: two *conditional ratio processes*:

$$q_1(f_1, t) = \frac{\phi_1(f_1, t)}{\phi_2(1, t)} \quad \text{and} \quad q_2(f_2, t) = \frac{\phi_2(f_2, t)}{\phi_1(1, t)}$$

The *Bayes Factors:* (the ratio of two integrated likelihoods)

$$B_{12} = \frac{\phi_1(1, t)}{\phi_2(1, t)} = q_1(1, t) \quad \text{and} \quad B_{21} = \frac{\phi_2(1, t)}{\phi_1(1, t)} = q_2(1, t)$$

- *Strongly Reject Model 1* if BF_{21} is larger than 20.
- *Decisively Reject Model 1* if BF_{21} is larger than 150.

Posterior Model Probabilities

Suppose we are comparing r models: $M_1, M_2, \cdots, M_r$.
Let $P(M_i|\mathcal{F}_0^{Y,V})$ be the prior probability of Model i.
Define: the *posterior model probability* as

$$P(M_i|\mathcal{F}_t^{Y,V}) = \frac{P(M_i|\mathcal{F}_0^{Y,V})\phi_i(1,t)}{\sum_{j=1}^r P(M_j|\mathcal{F}_0^{Y,V})\phi_j(1,t)} = \Big[\sum_{j=1}^r \frac{P(M_j|\mathcal{F}_0^{Y,V})}{P(M_i|\mathcal{F}_0^{Y,V})} B_{ji}(t)\Big]^{-1}.$$

- If $P(M_j|\mathcal{F}_0^{Y,V}) = 1/q$, for $j = 1, 2, \cdots, r$, then

$$P(M_i|\mathcal{F}_t^{Y,V}) = \frac{\phi_i(1,t)}{\sum_{j=1}^r \phi_j(1,t)} = \Big[\sum_{j=1}^r B_{ji}(t)\Big]^{-1}.$$

- Select the model with highest posterior model probability.

Unnormalized Filtering Equation

- **Theorem 1:** (Zeng 2003, for Model I) *Under Assumptions 2.1–2.5, ϕ_t is the unique measure-valued solution of the following SPDE, called the unnormalized filtering equation:*

$$\phi(f,t) = \phi(f,0) + \int_0^t \phi(\mathbf{A}f,s)ds - \int_0^t \int_U \phi(f(\lambda(u)-1),s)\mu(du)ds$$

$$+ \int_0^t \int_U \phi(f(\lambda(u)-1),s-)Y(ds,du), \tag{5}$$

for every $t > 0$ and $f \in D(\mathbf{A})$ with $\lambda(u) = \lambda(u,s)$ given by (4).

Normalized Filtering Equation

Theorem 1: (continued) π_t *is the unique measure-valued solution of the following SPDE, called the normalized filtering equation:*

$$\pi(f,t) = \pi(f,0) + \int_0^t \pi(\mathbf{A}f,s)ds + \int_0^t \pi(f,s)\int_U \pi(\lambda(u),s)\mu(du)ds$$

$$-\int_0^t \int_U \pi(f\lambda(u),s)\mu(du)ds + \int_0^t \int_U \left[\frac{\pi(f\lambda(u),s-)}{\pi(\lambda(u),s-)} - \pi(f,s-)\right] dY(ds,du)$$

Remark: When $a(\theta^t, X^t, V^t, Z^t, t) = a(V^t, Z^t, t)$ (including *ACD model*), it can be simplified as: $\pi(f,t) =$

$$\pi(f,0) + \int_0^t \pi(\mathbf{A}f,s)ds + \int_0^t \int_U \left[\frac{\pi(fp(u),s-)}{\pi(p(u),s-)} - \pi(f,s-)\right] dY(ds,du)$$

where $p(u) = p(u|X(t);\theta(t),V^{t-},Z^{t-},t)$.

Evolution Equations for BF

- **Theorem 2:** (Kouritzin and Zeng 2005, for Model I) *Assume Model 1 has* $(\mathbf{A}^{\mathbf{1}}_{\mathbf{v}}, \lambda_1, \mu_1)$ *and Model 2 has* $(\mathbf{A}^{\mathbf{2}}_{\mathbf{v}}, \lambda_2, \mu_2)$. *Both models satisfy Assumptions 2.1–2.5. Then,* $(q_{1,t}, q_{2,t})$ *is the unique pair measure-valued solution of the following system of SPDEs,*

$$q_1(f_1,t) = q_1(f_1,0) + \int_0^t q_1(\mathbf{A_1} f_1, s)ds$$

$$+ \int_0^t \frac{q_1(f_1,s)}{q_2(1,s)} \int_U q_2(\lambda_2(u),s)\mu_2(du)ds - \int_0^t \int_U q_1(f_1\lambda_1(u),s)\mu_1(du)ds$$

$$+ \int_0^t \int_U \left[\frac{q_1(f_1\lambda_1(u),s-)}{q_2(\lambda_2(u),s-)} q_2(1,s-) - q_1(f_1,s-) \right] dY(ds,du)$$

and

$$q_2(f_2,t) = \ldots$$

A Consistency Theorem

- **Theorem 3:** (Zeng 2003, and Kouritzin and Zeng 2005, for Model I)
Suppose that Assumptions 2.1 to 2.5 hold for (θ, X, Y) *and* $(\theta_\epsilon, X_\epsilon, Y_\epsilon)$*.*
If $(\theta_\epsilon, X_\epsilon) \Rightarrow (\theta, X)$ *as* $\epsilon \to 0$*, then for bounded continuous functions,* f*,*
(i) $Y_\epsilon \Rightarrow Y$*,* *(ii)* $\phi_\epsilon(f,t) \Rightarrow \phi(f,t)$*,* *(iii)* $\pi_\epsilon(f,t) \Rightarrow \pi(f,t)$*.*
In the two-model case for model selection, then
(iv) $q_{k,\epsilon}(f_k,t) \Rightarrow q_k(f_k,t)$ *for* $k = 1, 2$ *simultaneously.*
In the r-model case for model selection, then
(v) $P_\epsilon(M_i|\mathcal{F}_t^{Y,V}) \Rightarrow P(M_i|\mathcal{F}_t^{Y,V})$*, for* $i = 1, 2, \cdots r$ *simultaneously.*

Sketch of Proof:

- First, use Kurtz and Protter (1996)'s theorem on convergence of semi-martingale random measure and the Continuous Mapping theorem to prove $L_\epsilon \Rightarrow L$. Then, $((\theta_\epsilon, X_\epsilon), Y_\epsilon, L_\epsilon) \Rightarrow ((\theta, X), Y, L)$.
- Second, use Goggin (1994)'s or Kouritzin and Zeng (2005)'s theorems on convergence of conditional expectations and the Continuous Mapping theorem to prove (ii), (iii), (iv) and (v).

Markov Chain Approximation Method

Construction of Parallelizable Recursive Online Algorithms – *for computing nearly posterior, likelihoods, Bayes factors and posterior model probabilities*

For Example, to compute the *nearly* posterior: **Three Steps:**

- Construct a Markov chain $(\theta_\epsilon, X_\epsilon)$ to approximate (θ, X).
- Derive the filtering (or evolution) equations for $(\theta_\epsilon, X_\epsilon, Y_\epsilon)$.
  - *Propagation Equation*:

$$\pi_\varepsilon(f, t_{i+1}-) = \pi_\varepsilon(f, t_i) + \int_{t_i}^{t_{i+1}-} \pi_\varepsilon(\mathbf{A}_\varepsilon f, s) ds.$$

  - *Updating Equation*:

$$\pi_\varepsilon(f, t_{i+1}) = \frac{\pi_\varepsilon(fp(u), t_{i+1}-)}{\pi_\varepsilon(p(u), t_{i+1}-)}$$

- Convert the equation for $(\theta_\epsilon, X_\epsilon, Y_\epsilon)$ to recursive algorithms by
  - (a) representing $\pi_\epsilon(\cdot, t)$, for example, as a finite array with components being $\pi_\epsilon(f, t)$ for lattice-point indicator f;
  - (b) approximating the time integral with an Euler scheme.

Review: Model for Treasury UHF Data

(Joint with D. Kuipers and X. Hu)

- **Intrinsic value process:**

$$\frac{dX_t}{X_t} = \mu dt + (\sigma + \kappa_1 V_1(t) + \kappa_2 V_2(t))dB_t$$

Generator: $\theta = (\mu, \sigma, \rho, \kappa_1, \kappa_2)$

$$\mathbf{A}_v f(x,\theta) = \mu x \frac{\partial}{\partial x} f(\theta, x) + \frac{1}{2}(\sigma + \kappa_1 v_1 + \kappa_2 v_2)^2 x^2 \frac{\partial^2}{\partial x^2} f(\theta, x).$$

where $V_1(t)$ is the indicator of buyer (+1) or seller (0) initiating;
$V_2(t)$ is the indicator of macroeconomic news period (+1), otherwise (0).

- **Trading Times:** An (Exponential or Weibull) ACD model.

Modeling Trading Noise

Price at time t_i : $\quad Z(t_i) = F(X(t_i)) = b_i(R[X(t_i), \frac{1}{M}] + D_i)$

- **Discrete noise:** $R[x, \frac{1}{M}]$, rounding function, where $M = 64, 128$.
- **Non-clustering noise:** $\{D_i\}$**,** has a doubly-geometric distribution:

$$P\{D = d\} = \begin{cases} 1-\rho & \text{if } d = 0 \\ \frac{1}{2}(1-\rho)\rho^{M|d|} & \text{if } d = \pm\frac{i}{M} \text{ for } i = 1, 2, 3.... \end{cases}.$$

- **Clustering noise:** $b_i(\cdot)$**,** a random biasing function

biasing rule: Set $z' = R[X(t_i), \frac{1}{M}] + D_i$ and $z = Z(t_i) = b(z')$.

· If fractional part of z' is an even Mth, z stays on z' w. p. 1.

· If the fractional part of z' is an odd Mth, then

z' moves to the closest $M/2$th w.p. α,

or z' moves to the closest odd $M/4$th or integer w.p. β,

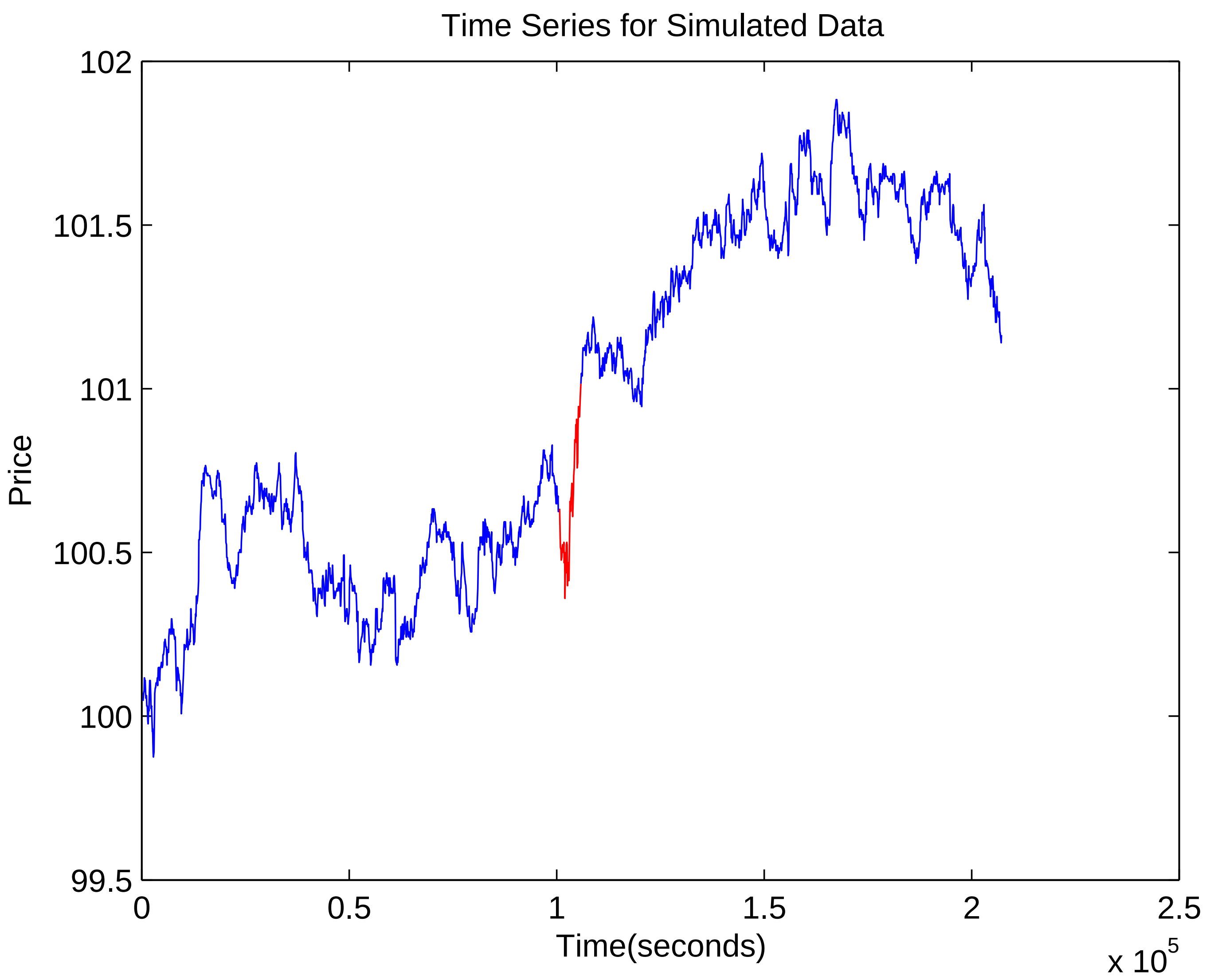
or z stays on z' w.p. $1-\alpha-\beta$.

- Parameters in the Model : $(\mu, \sigma, \rho, \kappa_1, \kappa_2, \alpha, \beta)$ + ACD's.

Consistency of Bayes Estimates

- **Theorem 4:** *For the simple filtering model of GBM with factors described above, when the clustering parameters* (α, β) *are known, and* $\theta = (\mu, \sigma, \rho, \kappa_1, \kappa_2)$ *has a prior, then, the Bayes estimates are consistent almost surely. Namely,* $E[f(\theta)|\mathcal{F}_t^{Z,V}] \to f(\theta)$ *as* $t \to \infty$ *for bounded continuous function* f*.*

Simulated Data



Bayes Est. for a Simulated Data Set

Table 1: Bayes estimates for 2050 simulated data when 50 data have extra volatility,

Models	μ	σ	ρ	κ_2
True	5.00E-8	2.00E-5	0.05	3.00E-5
Bayes Est.	4.27E-8	2.00E-5	0.0477	2.86E-5
	(1.08E-8)	(2.89E-14)	(0.0056)	(6.58E-6)

Estimation Results

Table 2: Annualized Bayes estimates for 10yr Treasury UHF Data, 8/15/-12/31, 2000

Models	μ	σ	ρ	κ_1	κ_2
Case I:	32.77%	5.04%	0.0538	-0.39%	n.a.
only κ_1	(4.58%)	(0.11%)	(0.0040)	(0.05%)	(n.a.)
Case II:	13.60%	4.86%	0.0477	n.a.	3.87%
only κ_1	(7.34%)	(0.05%)	(0.0044)	n.a.	(0.29%)
Case III:	33.71%	5.28%	0.0273	-0.59%	3.80%
both κ_1, κ_2	(3.56%)	(0.02%)	(0.0043)	(0.04%)	(0.27%)

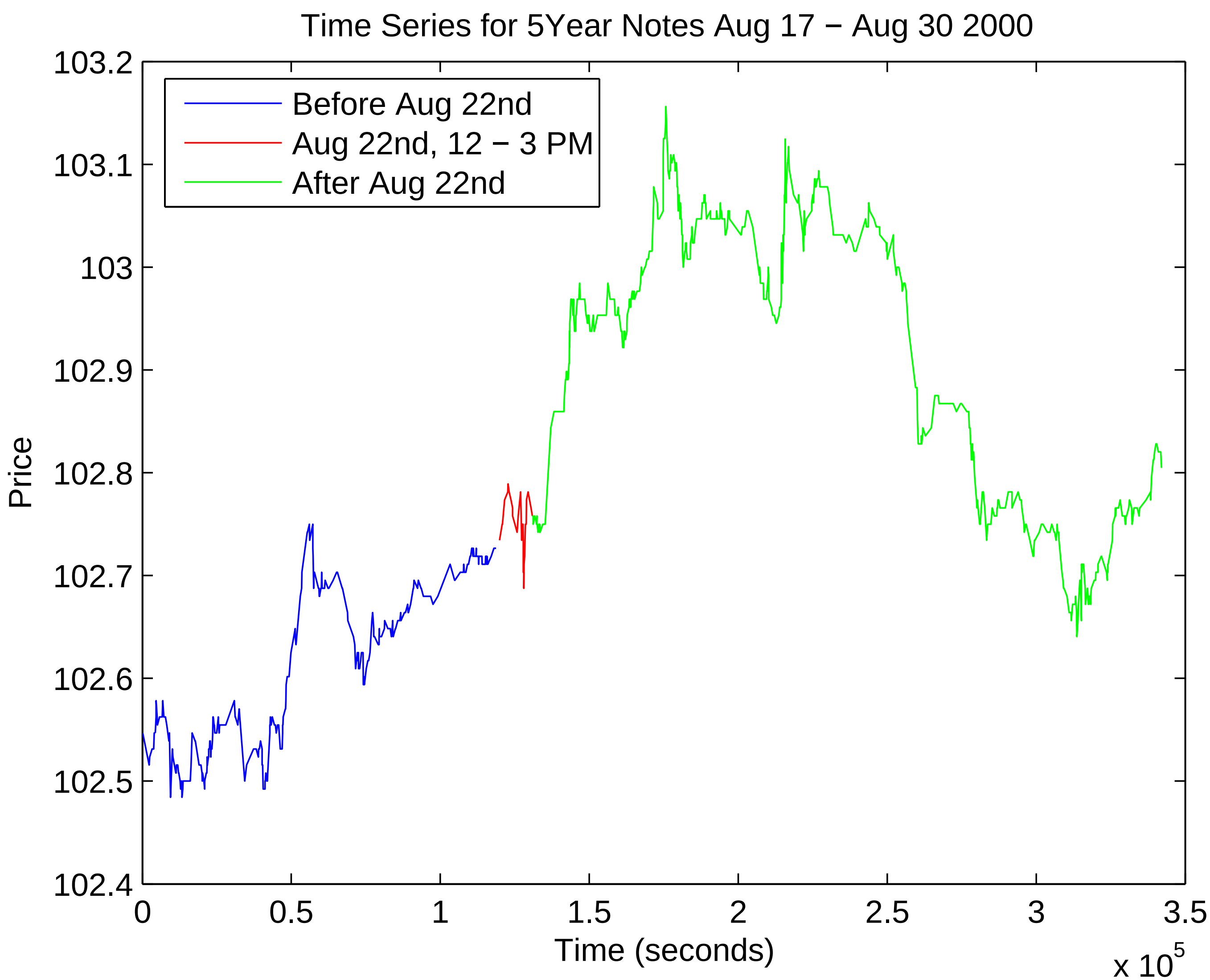
- These results are consistent with Fleming and Remolona (1999), Green (2004) and Brandt and Kavacejz (2004): Primary impact: Economic News; Secondary impact: order flow and liquidity.

Conclusions and Future Works

- Financial applications on market microstructure theory
- Exponential α-stable process as $X(t)$
- Heston's model as $X(t)$
- Numerical efficient algorithms.
- Statistics and information theory:
  - Consistency, CLT for the estimators of parameters.
  - Mutual information and its rate and its applications
- Mathematical finance:
  - Option pricing and hedging, portfolio optimization, and utility maximization.
- Applications to algorithmic trading?

Related papers, real data examples, Fortran codes are available at `http://mendota.umkc.edu/paper-tick.html`

Real Data: FOMC Period I



Real Data: FOMC Period II