Regime Shifts and the Term Structure of Interest Rates

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1 Introduction

Since the seminal paper of Duffie and Kan (1996), most empirical research on the term structure of interest rates have focused on a class of linear models, generally referred to as "affine term structure models". In this class of models, the yield on a τ-period bond (free of default risk), $i_{\tau,t}$, can be obtained as a linear function of some underlying state variables, X_t

$$i_{\tau,t} = A_\tau + B_\tau' X_t$$

where the coefficients A_τ and B_τ are determined by a system of differential equations. Since these models produce such a closed-form solution for the entire yield curve, they become very tractable in empirical applications. Piazzesi (2003) and Dai and Singleton (2003) provide excellent surveys of affine term structure models.

Non-linearity can be introduced into dynamic models of the term structure of interest rates either by generalizing the affine specification to a quadratic form along the line of Ahn, Dittmar, and Gallant (2002), or by including a Poisson jump component as an additional state variable as in Ahn and Thompson (1998), Das (2002) or Piazzesi (2005) among others. Another approach is to incorporate Markov regime shifts into an otherwise standard affine model. Bansal and Zhou (2002), Wu and Zeng (2005), Dai, Singleton and Yang (2007) are some recent examples. These non-linear models not only are more general than models without regime shifts, but also retain much of the tractability of the standard affine models. Moreover, there are natural economic interpretations of regime shifts. For example, much documented empirical evidence shows that the aggregate economy experiences recurrent shifts between distinct regimes of the business cycle. Such regimes shifts ought to be reflected into the dynamics of asset prices, and bond yields in particular. Another motivation for the regime-switching models is the impact of the monetary policy on interest rates. Most central banks in the world have now used some short-term interest rates as their policy instruments. A notable feature of the monetary policy behavior is that changes in the policy rate's target *of the same direction* are usually very persistent. For example, in the U.S. the Fed decreased its interest rate target 12 times consecutively between January 2001 and November 2002, and since June 2003 there have been 11 interest rate hikes by the Fed without a single decrease. Presumably such shifts in the overall monetary policy stance (from accommodative to tightening or vice versa) have more important effects on interest rates than a single interest rate change does. A model with regime shifts is the most convenient tool to capture such policy behavior.

In this chapter we survey some recent studies of dynamic models of the term structure of interest rates that incorporate Markov regimes shifts. In section 2 we summarize an early literature of regime-switching models that mainly focus on the short-term interest rate. The success of these models has since motivated fully-fledged dynamic asset pricing models for the whole yield curve under regime-switching. The next two sections attempt to summarize these recent studies. Section 3 considers regime-switching models in a discrete-time framework while section 4 contains continuous-time models. Section 5 provides some concluding remarks.

2 Regime-switching and Short-term Interest Rate

The misspecification of existing single-regime models of short-term interest rate has been widely discussed. One potential description for this mis-specified phenomenon is that the structural form of conditional means and variances is held fixed over the sample period. Ultimately, all such single-regime models, assuming that the short rate is mean-reverting, involve the estimation of a set of parameters that are assumed to be fixed throughout the entire sample period. However, the later discussed literature about the short-term interest rate in favor of a regime-switching model since it is more flexible and constitutes an attractive line in describing the 'style fact' of the short-term interest rate process. The regime-switching model is more attractive owing to its feature of incorporating significant nonlinearity in contract to the traditional linear property of the speed of revision and long-run mean inherent in most single-regime models. The regime-switching model is flexible enough to incorporate a different speed of revision to a different long-run mean at different times. The parameters in regime-switching model differing in different regime can account for the possibility that the short rate DGP may undergo a finite number of changes over sample period, which can capture the stochastic behavior of time-varying short-term interest rates. Since the regimes are never observed and the parameters are unknown and have to be estimated, probabilistic statements can be made about the relative likelihood of their occurrence, conditional on an information set.

The period of unprecedented interest rate volatility always coincides with changes in business cycle caused by various economic or non-economic shocks. In general, changes in monetary or fiscal policies result in a business cycle fluctuation, which may cause interest rates to behave quite differently in different time periods. The real world has also experienced various shocks in the economic environment within past few decades. For examples, the 1973 OAPEC oil crisis, the October 1987 stock market crash, 1997 Asian financial crisis, 2000 dot com crash, 2001 911-event, 2007 subprime mortgage crisis… and so on.[9] Since the stochastic behavior of short-term interest rates is well-described by regime-switching models, the earlier literature of regime-switching models applied to the interest rate process mainly focuses on the short-term interest rate. Those researches addressed on the earlier regime-switching models of the short-term interest rate follow the classical paper by Hamilton (1988). In this strand of literature, the models are mainly about the short-term interest rate alone, and long-term rates are usually related to the short-term rate via the expectation hypothesis. Examples include Lewis (1991), Cechetti, Lam and Mark (1993), Evans and Lewis (1995), Sola and Driffill (1994), Garcia and Perron (1996), Gary (1996), Bekaert, Hodrick and Marshall (2001), Ang and Bekaert (2002) among others. Most of the past literature has estimated the two-state regime-switching models, with the exception of Garcia and Perron (1996) and Bekaert, Hodrick and Marshall (2001) that focused mainly on the three-state models.

2.1 Short-term interest rate models

Short-term interest rate models should capture two well-known properties of the

[9] OAPEC is an abbreviation for Organization of Arab Petroleum Exporting Countries consisting of the Arab members of OPEC plus Egypt and Syria.

short rate process, mean-revision and leptokurtic unconditional distribution. Two most common classes of short rate models are known as diffusion models and GARCH (general autoregressive conditional heteroskedasticity) models.[10] We review the diffusion model first in this paper and construct a regime-switching model later followed the framework of the diffusion model.

In continuous time or diffusion models, the short term interest rate is usually described as based on the Brownian motion. The dynamics of the short rate is thus expressed by the stochastic differential equation as the following framework.

$$dr = (\alpha + \beta r)dt + \sigma\sqrt{r}\,dW \qquad (1)$$

where dW is the increment from a standard Brownian motion.

This stochastic differential equation is usually transferred to an autoregressive (AR) model for an estimation purpose.[11]

$$r_t = \alpha + \beta r_{t-1} + \varepsilon_t \qquad (2)$$

where $E[\varepsilon_t / \Phi_{t-1}] = 0$ and $E[\varepsilon_t^2 / \Phi_{t-1}] = \sigma^2 r_{t-1}^{2\gamma}$. Φ_{t-1} is the agents' information set at time t-1. From this equation, mean revision and leptokurtosis can be captured by setting $\beta < 0$ and $\gamma > 0$ (conditional heteroskedasticity), respectively.

2.2 Regime-switching

Regime-switching can be viewed as the state changes in a finite Markov chain.

For a general model considered in Gray (1996), we specify a special case that allows the short rate regime-switching setting held with regime-switching mean and variance.

$$r_t = \alpha_{s_t} + \beta_{s_t} r_{t-1} + \varepsilon_t \ , \qquad \varepsilon_t \sim N(0,\ \sigma_{s_t}^2) \qquad (3)$$

where s_t is the unobserved state variable presumed to follow a two-state Markov chain with transition probability, p_{ij}. β_{s_t} is the parameter for the measurement of the speed of revision to long-run mean in state s_t. The error term, ε_t, follows a normal distribution with 0 mean and a standard deviation of σ_{s_t} in state s_t.

Equation (3) is thought to follow a regime-switching framework by quasi-maximum likelihood as described in Hamilton (1989). The testable scheme is expressed as follows.

$$\alpha_{S_t} = \begin{cases} \alpha_1 & if \quad s_t = 1 \\ \alpha_2 & if \quad s_t = 2 \end{cases} \qquad \beta_{S_t} = \begin{cases} \beta_1 & if \quad s_t = 1 \\ \beta_2 & if \quad s_t = 2 \end{cases} \qquad \sigma_{S_t} = \begin{cases} \sigma_1 & if \quad s_t = 1 \\ \sigma_2 & if \quad s_t = 2 \end{cases}$$

The evolution of the unobservable state variable is assumed to follow a two-state first-order Markov chain process satisfying $p_{11} + p_{12} = p_{21} + p_{22} = 1$, where

[10] Engle (1982) shows that a possible cause of the leptokurtosis in the unconditional distribution is conditional heteroskedasticity.

[11] See from Chan et al. (1992) and Gray (1996) for more detail.

$p_{ij} = \Pr(s_t = j / s_{t-1} = i)$ gives the probability that state i followed by state j.[12] The state in each time point determines which of the two normal densities is used to generate the model. For our case of short-term interest rate, it is assumed to switch between two regimes (different long-run mean and speed of revision) according to transition probabilities.

Quasi-Maximum Likelihood Estimation of parameters

There are various ways to estimate the regime-switching model.[13] The estimation of the MS setting of equation (3) mainly follows Garcia and Perron (1996), which employs Hamilton's (1989) Markov-switching estimation by quasi-maximum likelihood.[14]

Let $y_t = R_t$, $x_t = (1, V_t)'$ and $\delta_{s_t} = (\alpha_{s_t}, \beta_{s_t})$. Equation (3) can be expressed as:[15]

$$y_t = x_t^{'}\delta_{s_t} + \varepsilon_t \quad , \qquad \varepsilon_t \sim N(0, \ \sigma_{s_t}^2)$$

This regime-switching model assumes that the variance is also shifting between regimes. s_t is the unobserved state variable presumed to follow a two-state Markov chain with transition probability, p_{ij}.

As usual, we use capital letters X_t and Y_t to represent all the information available up to time t and θ to denote the vector of the unknown population parameters.

i.e., $X_t = (x_1, x_2, \ \ldots \ , x_t)^{'}$, $Y_t = (y_1, y_2, \ \ldots \ , y_t)^{'}$ and $\theta = (\theta_1^{'}, \theta_2^{'})^{'}$, where X_t is exogenous or predetermined and conditional on s_{t-1}, and s_t is independent of X_t. The grouping parameter vector can be decomposed by $\theta_1 = (\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma_1, \sigma_2)^{'}$ and $\theta_2 = (p_{11}, p_{22})^{'}$.

By denoting $\tilde{X}_T = (x_1, x_2, \ \ldots \ , x_T)^{'}$, $\tilde{Y}_T = (y_1, y_2, \ \ldots \ , y_T)^{'}$, and $\tilde{S}_T = (s_1, s_2, \ \ldots \ , s_T)^{'}$, the joint density of $\tilde{Y}_T$ and $\tilde{S}_T$ is as:

$$\begin{aligned} f(\tilde{Y}_T, \tilde{S}_T; \tilde{X}_T, \theta) &= f(\tilde{Y}_T / \tilde{S}_T; \tilde{X}_T, \theta_1) \ \times \ f(\tilde{S}_T; \tilde{X}_T, \theta_2) \\ &= \prod_{t=1}^{T} f(y_t / s_t; x_t, \theta_1) \ \times \ \prod_{t=1}^{T} f(s_t / s_{t-1}; x_t, \theta_2) \end{aligned}$$

The log likelihood function can thus be expressed as:

$$\ln(\tilde{Y}_T, \tilde{S}_T; \tilde{X}, \theta) = \sum_{t=1}^{T} \ln[f(y_t / s_t; x_t, \theta_1)] \ \times \ \sum_{t=1}^{T} \ln[f(s_t / s_{t-1}; x_t, \theta_2)]$$

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[12] Markov property argues that the process of s_t depends on the past realizations only through s_{t-1}.

[13] see Kim and Nelson (1999)

[14] Garcia and Perron (1996) employs Hamilton's (1989) regime-switching model to explicitly account for regime shifts in an autoregressive model with three-state regime-switching mean and variance.

[15] For simplicity, the following analysis is based only on one industry. We thus omit the symbol i.

If the state is known, the parameter vector θ_2 would be irrelevant and the log likelihood function would be maximized with respect to θ_1.

$$\frac{\partial \ln[f(\tilde{Y}_T, \tilde{S}_T; \tilde{X}_T, \theta)]}{\partial \theta_1} = \sum_{t=1}^{T} \frac{\partial \ln[f(y_t / s_t; x_t, \theta_1)]}{\partial \theta_1}$$

Now, let's introduce the estimation produces as the following way.

Filter probability

We assume that θ is already observed.[16] Based on Hamilton (1994), the derivation begins with the unconditional probability of the state of the first observation.

$$p(s_t = 1) = \frac{1 - p_{22}}{(1 - p_{11}) + (1 - p_{22})} = \gamma \text{ , and consequently } p(s_t = 2) = 1 - \gamma$$

Given Y_{t-1}, the joint probability of s_{t-1} and s_t is:

$$\begin{aligned} p(s_t, s_{t-1} / Y_{t-1}; X_t) &= p(s_t / s_{t-1}, Y_{t-1}; X_{t-1}) \times p(s_{t-1} / Y_{t-1}; X_{t-1}) \\ &= p(s_t / s_{t-1}) p(s_{t-1} / Y_{t-1}; X_{t-1}) \end{aligned} \quad (4)$$

where the first equality is given by Bayes' theorem and the second one is by the independence principle of the Markov chain. Since the transition probability $p(s_t / s_{t-1})$ and the filter probability at time t-1, $p(s_{t-1} / Y_{t-1}; X_{t-1})$, are both known at time t, it is not difficult to calculate $p(s_t, s_{t-1} / Y_{t-1}; X_t)$ in Equation (4).

Summing up s_{t-1} to get the conditional marginal distribution of s_t:

$$p(s_t / Y_{t-1}; X_t) = \sum_{s_{t-1}=1}^{2} p(s_t, s_{t-1} / Y_{t-1}; X_t) \quad (5)$$

The joint probability of y_t and S_t at time t is then calculated as:[17]

$$p(y_t, s_t / Y_{t-1}; X_t) = f(y_t / s_t, Y_{t-1}; X_{t-1}) \times p(s_t / Y_{t-1}; X_{t-1}) \quad (6)$$

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16 The following derivations are mainly from Shen (1994).

17 f(.) and p(.) denote the continuous and discrete density function, respectively.

The first term on the right hand side is the sample likelihood function and the second term is from Equation (5), so Equation (6) can be calculated. Therefore, the filter inference about the probable regime at time t is given by:

$$P(s_t/Y_t;X_t) = \frac{p(y_t, s_t/Y_{t-1};X_t)}{p(y_t/Y_{t-1};X_t)}$$
$$= \frac{f(y_t/s_t, Y_{t-1};X_t)p(s_t/Y_{t-1};X_t)}{\sum_{s_t=1}^{2} f(y_t/s_t, Y_{t-1};X_t)p(s_t/Y_{t-1};X_t)}$$

Smoothed probability

The derivation of filter probability utilizes the information up to time t. Alternatively, we can use the full sample of *ex post* available information to draw the inference. It is therefore more efficient in the sense that all the information up to time T is utilized instead of t. Similarly, the smoothed probability of the first observation has to be derived. Consider the joint probability of y_t, s_t and s_1:

$$p(y_t, s_t, s_1/Y_{t-1};X_t) = f(y_t/s_t, s_1, Y_{t-1};X_t) \times p(s_t, s_1/Y_{t-1};X_{t-1})$$
$$= f(y_t/s_t, s_1, Y_{t-1};X_t) \times p(s_t/s_1) \times p(s_1/Y_{t-1};X_{t-1}) \qquad (7)$$

where the first equality is given by Bayes' theorem and the second one is by the Bayes' theorem and independence principle of Markov chain. The first term on the right hand side of Equation (7) is the sample likelihood function. The second term can be derived by:

$$p(s_t/s_1) = \sum_{s_t=1}^{2} p(s_t, s_{t-1}/s_1)$$
$$= \sum_{s_t=1}^{2} p(s_t/s_{t-1})p(s_{t-1}/s_1)$$
$$= \sum_{s_{t-1}=1}^{2}\sum_{s_{t-2}=1}^{2} p(s_t/s_{t-1})\, p(s_{t-1}/s_{t-2})\, p(s_{t-2}/s_1)$$

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$$= \sum_{s_{t-1}=1}^{2}\sum_{s_{t-2}=1}^{2} \ldots\ldots\ldots\ldots\ldots\ldots \sum_{s_2=1}^{2} p(s_1/s_{t-1})$$
$$\times \quad p(s_{t-1}/s_{t-2}) \ldots \quad p(s_2/s_1)$$

and the third term is the filter probability at time t-1. These terms are all known at time t and can be used to calculate Equation (7).

The joint probability of s_t and s_1 is thus given by:

$$P(s_t, s_1 / Y_t ; X_t) = \frac{p(y_t, s_t, s_1 / Y_{t-1} ; X_t)}{p(y_t / Y_{t-1} ; X_t)}$$

The numerator is from Equation (7) and the denominator can be derived by summing up s_t and s_1 of Equation (7). The conditional marginal probability of s_1 is then given by summing up s_t:

$$p(s_1 / Y_t ; X_t) = \sum_{s_t=1}^{2} p(s_t, s_1 / Y_t ; X_t)$$

and this is the smoothed probability of the first observation at time t. Similarly, the smoothed probability at time t+1 can be obtained by:

$$P(s_{t+1}, s_1 / Y_{t+1} ; X_{t+1}) = \frac{p(y_{t+1}, s_{t+1}, s_1 / Y_t ; X_{t+1})}{p(y_{t+1} / Y_t ; X_{t+1})}$$

where,

$$\begin{aligned} p(y_{t+1}, s_{t+1}, s_1 / Y_t ; X_{t+1}) &= f(y_{t+1} / s_{t+1}, s_1, Y_t ; X_{t+1}) \\ &\quad \times p(s_{t+1}, s_1 / Y_t ; X_{t+1}) \\ &= f(y_{t+1} / s_{t+1}, s_1, Y_t ; X_{t+1}) \\ &\quad \times p(s_{t+1} / s_1) \, p(s_1 / Y_t ; X_{t+1}) \end{aligned}$$

and

$$\begin{aligned} p(s_{t+1} / s_1) = \sum_{s_{t-1}=1}^{2} \sum_{s_{t-2}=1}^{2} \dots\dots\dots\dots \sum_{s_2=1}^{2} p(s_1 / s_{t-1}) \\ \times \quad p(s_{t+1} / s_t) \, p(s_t / s_{t-1}) \dots \quad p(s_2 / s_1) \end{aligned}$$

Summing up s_{t+1} to get the smoothed probability of the first observation at time t+1 yields:

$$p(s_1 / Y_{t+1} ; X_{t+1}) = \sum_{s_{t+1}=1}^{2} p(s_{t+1}, s_1 / Y_{t+1} ; X_{t+1})$$

By repeating the above steps, we are able to get the smoothed probability of the first observation at time T:

$$p(s_1 / Y_T ; X_T) = \sum_{s_T=1}^{2} p(s_T, s_1 / Y_T ; X_T)$$

Similarly, the smoothed probability of the ith observation at time T is given by:

$$p(s_t/Y_T\,; X_T) = \sum_{s_T=1}^{2} p(s_T, s_t/Y_T\,; X_T), \qquad t = 1,2,\ldots.., T$$

Estimation

According to the smoothed probability derived in the previous section, we can say that the observations were generated from the first state with probability

p(s_t = 1 | Y_T ;X_T) and from the second state with probability p(s_t = 2 | Y_T ;X_T).

Hamilton (1994) shows the relevant conditions of the maximum likelihood estimates of the ϕ_{s_t} are:

$$\sum_{t=1}^{T}(y_t = x'_t\hat{\phi}_j)x_t \times p(s_t \mid Y_t; X_t) = 0\,, \qquad j = 1,2 \qquad (8)$$

$$\hat{\sigma}^2 = \frac{\sum_{t=1}^{T}\sum_{j=1}^{2}(y_t - x'\hat{\phi}_j) \times p(s_t \mid Y_t; X_t)}{T} \qquad (9)$$

Equation (8) implies that $\hat{\phi}_j$ satisfies a weighted OLS orthogonality condition where each observation is weighted by the probability that it came from regime j. In particular, $\hat{\phi}_j$ can be found from an OLS regression of $y_t^*(j)$ on $x_t^*(j)$:

$$\hat{\phi}_j = \left[\sum_{t=1}^{T} x_t^*(j)x_t^*(j)'\right]^{-1}\left[\sum_{t=1}^{T} x_t^*(j)y_t^*(j)'\right], \qquad j = 1,2$$

where,

$$y_t^*(j) = y_t \times \sqrt{p(s_t = j \mid Y_T; X_T)}$$

$$x_t^*(j) = x_t \times \sqrt{p(s_t = j \mid X_T; Y_T)}$$

where j denotes the present state and " * " is used to distinguish the terms of the weighted observations from the original observations.

The estimate of σ^2 in Equation (9) is just the combined sum of the squared residuals from these two regressions divided by T.

Hamilton (1994) also shows the maximum likelihood estimates for the transition probabilities:

$$p_{ij} = \frac{\sum_{t=2}^{T} p(s_t = j, s_{t-1} = j \mid Y_T; X_T)}{\sum_{t=2}^{T} p(s_{t-1} = i \mid Y_T; X_T)}$$

which is essentially the number of times state i followed by state j divided by the number of times the process was in state ***i*** .

3 Regime-switching Term Structure Models in Discreet Time

Instead of focusing on a single interest rate, dynamic models of the term structure of interest rates attempt to model the joint movements of interest rates across the whole spectrum of maturities. The standard approach begins by first postulating a set of state variables, denoted as X_t, that underlie the dynamics of interest rates. Since under the no-arbitrage condition, a positive stochastic discount factor, denoted as *Mt*, exists and determines all bond prices, we can make further parametric assumptions about M_t, and the market price of risk in particular. Affine models of the term structure of interest rates are obtained after assuming both the short-term interest rate and the market price of risk are linear functions of X_t.[18]

The main advantage of this class of models is their tractability. Because the solution of the term structure of interest rates can be obtained as a linear function of the state variable X_t, which makes it easy to implement the models empirically. In this section, we generalize the standard affine models by incorporating Markov regime shifts in the dynamics of X_t and M_t. We can see that regime shifts not only make an otherwise standard model more flexible, they also introduce a new regime-switching risk premium in addition to the risk premiums due to shocks to the state variable X_t.

3.1 State variables

Let's assume that there are K possible regimes, and let S_t denote the regime at time *t*. S_t follows a Markov-switching process with transition probability $\pi(S_t, S_{t+1})$. For example, if $S_t = s, S_{t+1} = s^{'}, \pi(S_t, S_{t+1})$ gives the probability of switching from regime *s* at time *t* to regime $s^{'}$ at time *t+1*.

Let X_t be an $N \times 1$ vector that contains all other state variables. We assume that X_t follows a stationary mean-reverting process conditional on each regime, that is

$$X_t = \mu(S_t) + \Phi(S_t)X_{t-1} + \sum(X_{t-1}, S_t)\varepsilon_t \qquad (10)$$

where ε_{t+1} is an $N \times 1$ standard normal random variable, $\mu(S_t)$ and $\Phi(S_t)$ are regime-dependent $N \times 1$ vector and $N \times N$ matrix respectively, and $\sum(X_{t-1}, S_t)$ is an $N \times N$ diagonal matrix given by[19]

[18] In the case of stochastic volatility, affine models assume that the product of the market price of risk and the volatility term is a linear function of the state variable X_t. In other words, it is assumed that, under the risk-neutral probability measure, X_t follows a linear mean-reverting process

[19] Of course some regularity conditions need to be imposed on the parameters so that the term inside the square root is non-negative and the process is well defined. See Dai, Le and Singleton (2006) for more details.

$$\sum(X_{t-1}, S_t) = \begin{pmatrix} \sqrt{\sigma_{0,1}(S_t) + \sigma_{1,1}^{'}(S_t)X_{t-1}} & & \\ & \ddots & \\ & & \sqrt{\sigma_{0,N}(S_t) + \sigma_{1,N}^{'}(S_t)X_{t-1}} \end{pmatrix}$$

Again the presence of S_t indicates that the coefficients in Σ are regime-dependent. We can collect all the $\sigma_{0,i}(S_t)$ (i =1, ..., *N*) in an *N* × *N* diagonal matrix, $\sum_0(S_t)$, and collect all $\sigma_{1,i}(S_t)$ in an *N* × *N* matrix $\sum_1(S_t)$, with the *i*th column being $\sigma_{1,i}$. In some models such as Dai, Singleton and Yang (2007), the regime-dependence of the parameters is specified slightly differently. Parameters such as *μ*, *Φ* depend on the regime at time $t-1, S_{t-1}$, instead of the regime at time *t*. In other models such as those of Bansal and Zhou (2002) the parameters are assumed to depend on regime at time *t*, S_t, as in the current paper.

3.2 The stochastic discount factor

Under fairly general condition (see, for example, Harrison and Kreps, 1979), the absence of arbitrage in financial markets implies that there exists a positive stochastic discount factor, M_t, such that the price of an asset at time *t*, *Pt*, is given by

$$P_t = E_t(M_{t+1}X_{t+1})$$

where X_{t+1} is the random payoff of this asset at time *t + 1*. In the case of bonds, if we let $P_{n,t}$ denote the price of an *n*-period zero-coupon bond at time *t*, we have

$$P_{n,t} = E_t(M_{t+1}P_{n-1,t+1})$$

The model is completed by making specific assumptions about the stochastic discount factor, M_t, and the short-term interest rate. In particular, let

$$M_{t+1} = e^{-m_{t+1}} = e^{-i_t - \frac{1}{2}\lambda_t^{'}\lambda_t - \lambda_t^{'}\varepsilon_{t+1}}$$

where i_t is the (one-period) short-term interest rate, λ_t is the market price of risk and ε_{t+1} is the fundamental shocks that drive the state variables. Models differ in their assumptions about the market price of risk. For example, in Gaussian affine models, it is assumed that λ_t is a linear function of the state variable X_t.

Under regime shifts, we assume that

$$m_{t+1} = i_t + \frac{1}{2}\lambda_t^{'}(S_t, S_{t+1})\lambda_t(S_t, S_{t+1}) + \lambda_t^{'}(S_t, S_{t+1})\varepsilon_{t+1} \quad (11)$$

where $\lambda_t^{'}(S_t, S_{t+1})$ indicates that the market price of risk not only is a function of X_t, but also depends on regime S_t *and* S_{t+1}. As pointed out by Dai and Singleton (2003), the presence of S_{t+1} has an important implication. If λ_t depends *onl*y on S_t, it implies that a regime shift at time t + 1 will not have an impact on the stochastic discount factor, M_{t+1}, or investors' inter-temporal marginal rate of substitution. In order words, the regime-switching does not present a systematic risk to investors and hence will not be priced in the bond market. Assets are risky only because of the shock ε_{t+1}. What the regime-witching will do, however, is to make risk premiums, or the expected excess return on risky assets, regime-dependent. The risk premiums will be time-varying not only because they depend on X_t, but also because the parameters are regime-dependent. On the other hand, if λ_t depends on S_{t+1} as well, there will be an additional risk premium associated with regime-switching. Because in this case, regime shifts have a direct impact on M_{t+1} and will be regarded as a systematic risk just as the shock ε_{t+1}.

To obtain a closed-form solution for the term structure of interest rates, in general we need λ_t to satisfy

$$\sum_t \lambda_t(S_t, S_{t+1}) = \lambda_0(S_t, S_{t+1}) + \Lambda(S_t, S_{t+1}) X_{t+1} \tag{12}$$

or equivalently

$$\lambda_t(S_t, S_{t+1}) = \sum_t^{-1} [\lambda_0(S_t, S_{t+1}) + \Lambda(S_t, S_{t+1}) X_{t+1}] \tag{13}$$

where $\lambda_0(S_t, S_{t+1})$ is an *N × 1* vector of regime-dependent (on S_t and S_{t+1}) constant, $\Lambda(S_t, S_{t+1})$ is a regime-dependent (on S_t and S_{t+1}) *N × N* matrix. $\sum_t$ is the conditional volatility term of the state variable X_{t+1} as defined in (10).

The last component of a dynamic model of the term structure of interest rates is a specification of the short-term interest rate, i_t. Here, following the standard affine models, we assume that

$$i_t = A_1(S_t) + B_1^{'}(S_t) X_t \tag{14}$$

where $A_1(S_t)$ and $B_1(S_t)$ (an *N × 1* vector) are all regime-dependent parameters.

3.3 Solving for the term structure of interest rates

Again let $P_{n,t}$ being the price of a *n*-period zero-coupon bond at time *t*, we have

$$P_{n,t} = E_t(e^{-m_{t+1}} P_{n-1,t+1})$$

We guess the solution to the bond price is, for some regime-dependent coefficients $A_n(S_t)$ an $B_n(S_t)$,

$$P_{n,t} = e^{-A_n(S_t) - B_n^{'}(S_t) X_t}$$

Substituting into the asset pricing equation above, we obtain

$$e^{-A_n(S_t)-B_n^{'}(S_t)X_t} = E_t(e^{-i_t-\frac{1}{2}\lambda_t^{'}\lambda_t-\lambda_t^{'}\varepsilon_{t+1}} \times e^{-A_{n-1}(S_{t+1})-B_{n-1}^{'}(S_{n-1})X_{t+1}}) \qquad (15)$$

Note that $\lambda_t = \lambda_t(S_t, S_{t+1})$ depends on X_t, S_t and S_{t+1}. Since the regime-switching (from S_t to S_{t+1}) probability is given by $\pi(S_t, S_{t+1})$, (15) can be written as

$$e^{-A_n(S_t)-B_n^{'}(S_t)X_t} = \sum_{S_{t+1}} \pi(S_t, S_{t+1}) E(e^{\xi_{t+1}(S_t,S_{t+1})} \mid I_t, S_{t+1}) \qquad (16)$$

where I_t is the information set at time *t*, and ξ_{t+1} is given by

$$\xi_{t+1}(S_t, S_{t+1}) = [-i_t - \frac{1}{2}\lambda_t^{'}\lambda_t - \lambda_t^{'}\varepsilon_{t+1}] + [-A_{n-1}(S_{t+1}) - B_{n-1}^{'}(S_{t+1})X_{t+1}] \qquad (17)$$

First note that

$$E(e^{\xi_{t+1}(S_t,S_{t+1})} \mid I_t, S_{t+1}) = e^{E(\xi_{t+1}|I_t,S_{t+1})+\frac{1}{2}Var(\xi_{t+1}|I_t,S_{t+1})}$$

which is approximately[20]

$$E(e^{\xi_{t+1}(S_t,S_{t+1})} \mid I_t, S_{t+1}) \approx 1 + E(\xi_{t+1} \mid I_t, S_{t+1}) + \frac{1}{2}Var(\xi_{t+1} \mid I_t, S_{t+1})$$

Substituting back into (16) and using the same approximation for $e^{-A_n(S_t)-B_n^{'}(S_t)X_t}$,

we have

$$-A_n(S_t) - B_n^{'}(S_t)X_t = \sum_{S_{t+1}} \pi(S_t, S_{t+1})[E(\xi_{t+1} \mid I_t, S_{t+1}) + \frac{1}{2}Var(\xi_{t+1} \mid I_t, S_{t+1})] \qquad (18)$$

where ξ_{t+1} is given in (17). Equation (18) gives a system of difference equations for $A_n(S_t)$ and $B_n(S_t)$ that can be solved recursively, with the initial condition that $A_0(S_t) = 0$, $B_n(S_t) = 0$ for *all* S_t, and $A_1(S_t)$ and $B_1(S_t)$ are given in (14).

For example, if there are K possible regimes at each time *t*, (18) results in the following equations for $A_n(s)$ and $B_n(s)$ (s =1, 2, ..., *K*).

---

[20] This is the approximation used in Bansal and Zhou (2002).

$$B_n(s) = \sum_{s^*}^{K} \pi(s,s^*)[(\Phi - \Lambda(s,s^*))^{'} B_{n-1}(s^*) + \frac{1}{2}\sum\nolimits_1 B_{n-1}^2(s^*)] + B_1(s) \qquad (19)$$

$$A_n(s) = \sum_{s^*}^{K} \pi(s,s^*)[A_{n-1}(s^*) + B_{n-1}^{'}(s^*)(\mu - \lambda_0)(s,s^*)) - \frac{1}{2} B_{n-1}^{'}(s^*)\sum\nolimits_0 \sum\nolimits_0^{'} B_{n-1}(s^*) \quad (20)$$

Note that (19) and (20) define a system of 2 × K difference equations that must be solved jointly.

Denote $i_{n,t}$ as continuously compounding *n*-period interest rate, or the yield on the *n*-period bond. By definition,

$$P_{n,t} = e^{-ni_{n,t}}$$

Therefore the term structure of interest rates can be obtained as, for any maturity n and time t,

$$i_{n,t} = a_n(S_t) + b_n^{'}(S_t) X_t = \frac{A_n(S_t)}{n} + \frac{B_n(S_t)}{n} X_t \qquad (21)$$

In this model we have to rely on log-linear approximation to get an analytical solution to the terms structure of interest rates as in Bansal and Zhou (2002) because of the assumptions that the coefficient Λ in (12) and the coefficient B_1 in (14) are regime-dependent. Exact solutions can be obtained if we assume that both Λ and B_1 do not depend on regimes. This is essentially the assumption made in Dai, Singleton and Yang (2007). In their models, the factor loading coefficient B_n in the solution of the term structure is also independent of regimes. More general models can be obtained by making the regime-switching probability time-varying. That is we can assume that the conditional probability $\pi(S_t, S_{t+1})$ depends on X_t as in Dai, Singleton and Yang (2007). However, in this case, we need M_{t+1} to be separable in S_{t+1} and ε_{t+1} and place some additional restrictions on λ_t. This generalization can be more easily discussed in a continuous-time model below.

4 Regime-switching Term Structure Models in Continuous Time

Continuous-time models provide more analytical tractability. The affine term structure model of Duffie and Kan (1996) has been further studied in Dai and Singleton (2000) and generalized to "essentially affine" models in Duffie (2002) and "semi- affine " models in Duarte (2004). Das (2002) and Checko and Das (2002) incorporated jumps in an otherwise standard affine model. Cheridito, Filipovic and Kimmel (2007) considers more general specification of the market price of risk. Quadratic term structure models are studied in Ahn, Dittmar and Gallant (2002). See the survey paper by Dai and Singleton (2003) and see the book by Singleton (2006) for more recent development. This section focuses on the continuous-time term structure models with regime-shifting.

To the best of our knowledge, Landen (2000) provides the first continuous-time regime-switching model of the term structure of interest rates. Under the risk-neutral pricing measure, she derives the dynamics of the yield curve and obtains an explicit solution for a special regime-switching affine case. Dai and Singleton (2003) proposes a fully fledged dynamic model of the term structure of interest rates under regime-switching. Their model characterizes the dynamics of interest rates under both the risk-neutral and the physical measures in the presence of regime-switching risk premiums. Following the approach of Cox, Ingersoll and Ross (1985a,b) and Ahn and Thompson (1988), Wu and Zeng (2005) developed a regime-switching models from a general equilibrium framework under the systematic risk of regime shifts. Other studies of regime-switching term structure models include Elliott and Mamon (2001), Wu and Zeng (2004, 2007) and etc. Papers for regime-switching models of the term structure of interest rates under default risk can be found in Bielecki and Rutkowski (2000, 2001).

In the rest of this section, we present a slightly more general framework than the one in Wu and Zeng (2006). We first review a useful representation of regime shifts introduced in Wu and Zeng (2007) with a slight generalization. We then follow the no-arbitrage approach to develop a general tractable multi-factor dynamic model of the term structure of interest rates that includes not only regime shifts but also jumps. The model allows for regime-dependent jumps while both jump risk and regime-switching risk are priced. A closed form solution for the term structure is obtained for an affine-type model under log-linear approximations.

4.1 A useful representation for regime shift

Regime-shifting can be viewed as the state changes in a finite-state Markov chain. There are three commonly used mechanisms to model a Markov chain with time-varying transition probabilities. Each mechanism has its applications in the literature of interest rate term structure. The first is *Conditiona*l *Marko*v *Chai*n approach, which is discussed in the book of Yin and Zhang (1998) with many applications. Bielecki and Rutkowski (2000) and (2001) and Dai and Singleton (2003) are applications of this mechanism in modeling the term structure of interest rates. *Hidde*n *Marko*v *Mode*l (HMM) approach is the second mechanism, which is summarized in the book of Elliott et. al. (1995). Elliott and Mamon (2001) utilize the HMM approach to derive a model of the term structure of interest rates. The third is the *Marke*d *Poin*t *Proces*s or the *Rando*m *Measur*e approach, which is employed in Landen (2000). The mark space in Landen’s representation is $E = \{(i, j) : i \in \{0,1,..., N\}, j \in \{0,1,..., N\}, i \neq j\}$, the product space of regimes including all possible regime switching. The third mechanism has an important advantage over the previous two mechanisms. Namely, the regime process has a simple integral form so that Ito’s formula can be applied easily.

Along the line of the third mechanism, a simpler integral form for the regime process is developed using only the space of regime as the mark space. This is introduced in Wu and Zeng (2007) and we discuss such representation here with a slight generalization.

Let S_t represent the most recent regime. There are two steps to obtain the simple integral representation of S_t.

First, we define the random counting measure $m(t,\cdot)$ for every $t > 0$.

.

Denote the mark space $U = \{0,1,...,N\}$ as all possible regimes with the power σ-algebra. Denote u as a generic point in U and A as a subset of *U*. We denote $m(t, A)$ as the cumulative number of entering a regime in A during the period (0,*t*]. For example, $m(t,\{u\})$ counts the cumulative times to shift to regime u during (0,*t*]. Then, m is the suitable random counting measure.

A marked point process or a random measure is uniquely characterized by its stochastic intensity kernel,[21] . To define the stochastic intensity kernel for $m(t,\cdot)$, we denote η as the usual counting measure on *U*. Note that η has these two properties: For $A \in U$, $\eta(A) = \int I_A \eta(du)$ (i.e. $\eta(A)$ counts the number of elements in A) and $\int_A f(u)\eta(du) = \sum_{u \in A} f(u)$. The stochastic intensity kernel can depend on the current time and regime. We also allow $m(t,\cdot)$ to depend on X_t, another state variable to be defined later.

We define the stochastic intensity kernel as

$$\gamma_m(dt, du) = h(u; X(t-), S(t-), t-)\eta(du)dt, \tag{22}$$

where $h(u; X(t-), S(t-), t-)$ is the conditional regime-shift (from regime $S(t-)$ to u) intensity at time t given $X(t-)$ and *S*(*t*-). Note that $h(u; X(t-), S(t-), t-)$ corresponds to the h_{ij} in the infinitesimal generator matrix of the Markov chain. We assume

$$h(u, X(t-), S(t-), t-) = e^{h_0(u, S(t-), t-) + h_1^{'}(u, S(t-), t-) X(t-)}$$

where $h_0(u, X(t-), S(t-), t-)$ is a real value and $h_1(u, X(t-), S(t-), t-)$ is a L $\times$ 1 vector. We choose $h(u, X(t-), S(t-), t-)$ as an exponential affine form so that we can obtain an approximate close form solution for the dynamics of the term structure. Intuitively, we can consider $\gamma_m(du, dt)$ as the conditional probability of shifting from Regime $S(t-)$ to Regime u during $[t, t + dt]$ given $X(t-)$ and $S(t-)$. We can express $\gamma_m(t, A)$, the compensator of $m(t, A)$, as

$$\gamma_m(t, A) = \int_0^t \int_A h(u; X(\tau-)S(\tau-), \tau-)\eta(du)d\tau = \sum_{u \in A} \int_0^t h(u; X(\tau-)S(\tau-), \tau-)d\tau$$

Second, we express $S(t)$ in the integral form below using the random measure defined above:

$$S(t) = S(0) + \int_{[0,t] \times U} (u - S(\tau-))m(d\tau, du) \tag{23}$$

[21] See Last and Brandt (1995) for detailed discussion of marked point process, stochastic intensity kernel and related results.

Note that $m(d\tau, du)$ takes two possible values: 0 or 1. It takes one only at the regime-switching time t_i and with $u = S(t_i)$, the new regime at time t_i. Hence, the above expression is but a telescoping sum: $S(t) = S(0) + \sum_{t_i<t}(S(t_i) - S(t_{(i-1)}))$. The corresponding differential form is:

$$dS(t) = \int_U (u - S(t-))m(dt, du) \tag{24}$$

To understand the above differential equation, we can assume there is a regime switching from $S(t-)$ to u occurs at time *t*, then $S(t) - S(t-) = u - S(t-)$ implying $S(u)$.

These two forms are crucial, because they allow the straightforward application of Ito's formula in the following subsections.

4.2 Regime-dependent jump diffusion model for the term structure of interest rates

This section proposes a general multi-factor term structure model under regime-switching jump diffusion. A closed form solution of the term structure is obtained for an affine-type model using log-linear approximation.

4.2.1 State variables

In this Section, the most recent regime $S(t)$ follows (23) or (24). Other L state variables, X_t, are described by the following equation, which is a continuous-time analogue of Equation (10) but includes an additional jump component:

$$dX_t = [\Theta_0(S_t, t) + \Theta_1(S_t, t)X_t]dt + \sum(X_t, S_t, t)dW_t + J(S_{t-}, t-)dN_t \tag{25}$$

where $\Theta_0(S_t, t)$ and $\Theta_1(S_t, t)$ are regime-dependent $L\times 1$ vector and $L\times L$ matrix respectively. $\sum(X_t, S_t, t)$ is an $L\times L$ diagonal matrix given by

$$\sum(X_t, S_t, t) = \begin{pmatrix} \sqrt{\sigma_{0,1}(S_t, t) + \sigma_{1,1}^{'}(S_t, t)X_t} & & \\ & \ddots & \\ & & \sqrt{\sigma_{0,L}(S_t, t) + \sigma_{1,L}^{'}(S_t, t)X_t} \end{pmatrix}$$

where $\sigma_{0,i}(S_t, t)$ (i =1, ..., *N*) are regime-dependent coefficients and all $\sigma_{1,i}(S_t, t)$ are regime-dependent $L\times 1$ vectors. W_t is a $L\times 1$ vector of independent standard Brownian motions; N_t is a $L\times 1$ vector of independent Poisson processes with $L\times 1$ time-varying and regime-dependent intensity $\delta_J(S_{t-},t-)$; $J(S_{t-},t-)$ is an $L\times L$ matrix of regime-dependent random jump size with a conditional density $g(J \mid S_{t-},t-)$. Given $\{S_{t-},t-\}$, we assume that $J(S_{t-},t-)$ are serially independent and are also independent of W_t and N_t.

4.2.2 The short rate

The instantaneous short-term interest r_t is a linear function of X_t given S_t and t

$$r_t = \varphi_0(S_t,t) + \varphi_1(S_t,t)' X_t \tag{26}$$

where $\varphi_0(S_t,t)$ is a scalar and $\varphi_1(S_t,t)$ is a $L\times 1$ vector.

4.2.3 The stochastic pricing Kernel

Using no arbitrage argument (see Harrison and Kreps 1979 for technical conditions), we further specify the pricing kernel, M_t, which is also called stochastic discount factor in Section 3.2. We first present it in a stochastic differential equation (SDE) form as

$$\frac{dM_t}{M_{t-}} = -r_t dt - \lambda_{D,t} dW_t - \Lambda'_{J,t-}(dN_t - \delta_J(S_{t-},t-)dt) - \int_U \lambda_S(X(t-),S(t-),t-)[m(dt,du) - \gamma_m(dt,du)]. \tag{27}$$

The $L\times 1$ vector of market prices of diffusion risk conditioning on X_t, S_t and t is

$$\lambda_{D,t} = \lambda_{D,t}(X_t,S_t,t) = \Lambda'_D(S_t,t)\sum(X_t,S_t,t)$$

where $\Lambda'_D(S_t,t) = (\lambda_{D,1}(S_t,t),\cdots\lambda_{D,L}(S_t,t))$ is an $L\times 1$ vector. The $L\times 1$ vector of market prices of jump risk conditioning on S_{t-} and $t-$ is

$$\lambda'_{J,t-} = \lambda'_J(S_{t-},t-) = (\lambda_{J,1}(S_{t-},t-),\cdots\lambda_{J,L}(S_{t-},t-))$$

And the market price of regime-switching (from regime $S(t-)$ to regime u) risk given $X(t-)$, $S(t-)$ and $t-$ is

$$\lambda_S(X(t-),S(t-),t-) = 1 - e^{\lambda_{0,S}(u,S(t-)) + \lambda'_{1,S}(u,(S(t-))X(t-)}$$

Where $$\lambda'_{1,S} = (\lambda_{1,S,1},\cdots\lambda_{1,S,L})$$

...

The specifications above complete the model for the term structure of interest rates, which can be solved by a change of probability measure. Specifically, for fixed $T > 0$, we define the equivalent martingale measure Q by the Radon-Nikodym derivative $\frac{dQ}{dP} = \xi_T / \xi_0$ where for $t \in [0,T]$

$$
\begin{aligned}
\xi_t = (e^{-\int_t^T r_\tau d\tau})(e^{-\int_0^t \lambda'_{D,\tau} dW(\tau) - \frac{1}{2}\lambda'_{D,\tau}\lambda_{D,\tau} d\tau}) \times \\
(e^{\int_0^t \lambda'_{J,\tau}\delta_{J,\tau} d\tau + \int_0^t \log(1-\lambda_{J,\tau})' dN_t}) \times \\
(e^{\int_0^t \int_U \lambda_S(u; X_{\tau-}, S_{\tau-}) \gamma_m(d\tau, du) + \int_0^t \int_U \log(1-\lambda_S(u; X_{\tau-}, S_{\tau-})) m(d\tau, du)})
\end{aligned} \tag{28}
$$

Observe that the first two terms of ξ_t correspond to the discrete time $M_{t+1} = e^{-m_{t+1}}$ where m_{t+1} is given in Equation (11). The extra two terms correspond to the market prices of jump and regime-shift.

In the absence of arbitrage, the price at time $t-$ of a default-free pure discount bond that matures at T , $P(t-,T)$ *, can be obtained as* ,

$$
P(t-,T) = E^Q_{t-}(e^{-\int_t^T r_\tau d\tau}) = E^Q\{e^{-\int_t^T r_\tau d\tau} \mid F_t\} = E^Q\{e^{-\int_t^T r_\tau d\tau} \mid X_{t-}, S_{t-}\} \tag{29}
$$

with the boundary condition $P(T-,T) = P(T,T)$ and the last equality comes from the Markov property of (X_t, S_t).

4.2.4 The dynamics under Q

To present the dynamics of state variables, X_t and S_t under equivalent martingale measure **Q**, we first define

$$
\textstyle\sum_0' = \sum_0'(S_t,t) = (\sigma_{0,1}(S_t,t), \cdots, \sigma_{0,L}(S_t,t))
$$

$$
\textstyle\sum_1 = \sum_1(S_t,t) = \begin{pmatrix} \sigma_{1,1}(S_t,t) & & \\ & \ddots & \\ & & \sigma_{1,L}(S_t,t) \end{pmatrix}
$$

and

$$
\Lambda_D = \Lambda_D(S_t,t) = \begin{pmatrix} \lambda_{D,1}(S_t,t) & & \\ & \ddots & \\ & & \lambda_{D,L}(S_t,t) \end{pmatrix}
$$

Then, the dynamics of X_t and S_t under **Q** are given by the following stochastic differential equations respectively

$$
dX_t = \tilde{\Theta}(X_t, S_t, t)dt + \textstyle\sum(X_t, S_t, t) d\widetilde{W}_t + J(X_{t-}, S_{t-}) d\widetilde{N}_t \tag{30}
$$

$$dS_t = \int_U \left(u - S_{t-}\right)\widetilde{m}\left(dt, du\right). \tag{31}$$

The drift term is

$$\tilde{\Theta}\left(X_t, S_t, t\right) = \left[\Theta_0\left(S_t, t\right) + \Theta_1\left(S_t, t\right) X_t\right] - \Sigma\left(X_t, S_t, t\right)^2 \lambda_D\left(S_t, t\right) \tag{32}$$

$$= \widetilde{\Theta}_0\left(S_t, t\right) + \widetilde{\Theta}_1\left(S_t, t\right) X_t \tag{33}$$

with

$$\widetilde{\Theta}_0\left(S_t, t\right) = \Theta_0\left(S_t, t\right) - \Lambda_D\left(S_t, t\right)\Sigma_0\left(S_t, t\right)$$

and

$$\widetilde{\Theta}_1\left(S_t, t\right) = \Theta_1\left(S_t, t\right) - \Lambda_D\left(S_t, t\right)\Sigma_1^{'}\left(S, t\right)$$

$\widetilde{W}_t$ is a L×1 standard Brownian motion under Q; $\widetilde{N}_t$ is a L×1 vector of Poisson processes with intensity $\tilde{\delta}_J\left(S_{t-}, t-\right)$ whose elements are given by $\tilde{\delta}_{i,J}\left(S_{t-}, t-\right) = \left[1 - \lambda_{i,J}\left(S_{t-}, t-\right)\right]\delta_{i,J}\left(S_{t-}, t-\right)$ for $i=1,\ldots,L$; $\widetilde{m}\left(t, A\right)$ is the marked point process with intensity matrix $\widetilde{H}\left(u; X_{t-}, S_{t-}\right) = \left\{\tilde{h}\left(u; X_{t-}, S_{t-}\right)\right\} =$ $\left\{h\left(u; X_{t-}, S_{t-}\right)\left(1 - \lambda_S\left(u; X_{t-}, S_{t-}\right)\right)\right\} = \left\{e^{\tilde{h}_0\left(u, S(t-)\right) + \tilde{h}_1^{'}\left(u, S(t-)\right) X_t}\right\}$ with $\tilde{h}_0\left(u, \left(S_{t-}\right)\right) =$ $h_0\left(u, S\left(t-\right)\right) + \lambda_{0,S}\left(u, S\left(t-\right)\right)$ and $\tilde{h}_1\left(u, S\left(t-\right)\right) = h_1\left(u, S\left(t-\right)\right) + \lambda_{1,S}\left(u, S\left(t-\right)\right)$. The compensator of $\widetilde{m}\left(t, A\right)$ under Q becomes

$$\gamma_{\widetilde{m}}\left(dt, du\right) = \left(1 - \lambda_S\left(u; X_{t-}, S_{t-}\right)\right)\gamma_m\left(dt, dz\right) = \tilde{h}\left(u; X_{t-}, S_{t-}\right)\eta\left(du\right)dt,$$

4.2.5 Bond Pricing

Let *P(t-,T) = F(t-,Xt-,St-,T) = F(t-, X, S, T)* where *X = Xt-* and *S = St-*. The following proposition gives the partial differential equation (PDE) determining the bond price.

Proposition 4.1 *The price of the default-free pure discount bond F (t-, X, S, T) defined in (29) satisfies the following partial differential equation*

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X^{'}}\widetilde{\Theta} + \frac{1}{2} tr\left(\frac{\partial^2 F}{\partial X \partial X^{'}}\Sigma\Sigma'\right)$$

$$+ E_{s,t-}^{Q}\left(\Delta_s F\right) + \tilde{\delta}_J^{'} E_{J,t-}^{Q}\left(\Delta_X F\right) = rF \tag{34}$$

with the boundary condition *F (T-, X, S, T)= F (T, X, S, T)* = 1. Note that $\Delta_S F = F\left(t-, X_{t-}, u, T\right) - F\left(t-, X_{t-}, S_{t-}, T\right)$ and $E_{S,t-}^{Q}\left(\Delta_S F\right) = \int_U \Delta_S F \tilde{h}\left(u; S_{t-}\right)\eta\left(du\right)$, i.e the mean of $\Delta_S F$ conditioning on X_{t-} and S_{t-} under Q;

$$\Delta_X F = \begin{pmatrix} F(t-, X_{t-}+J_1, S_{t-}, T) - F(t-, X_{t-}, S_{t-}, T) \\ \vdots \\ F(t-, X_{t-}+J_L, S_{t-}, T) - F(t-, X_{t-}, S_{t-}, T) \end{pmatrix}$$

and $E^Q_{J,t-}(\Delta_X F) = \int \Delta_X F \widetilde{g}(J \mid X_{t-}, S_{t-}) dJ$, i.e. the mean of $\Delta_X F$ conditioning on X_{t-} and S_{t-} under Q.

The above proposition follows from the Feynman-Kac formula and the Markov property of (X, S).

When we let $F(t-, X_{t-}, S_{t-}, T) = e^{A(S(t-),t-) + B'(S(t-),t-)X(t-)}$
with $A(S(t-),t-) = A(S(t-),t-,T)$ and $B(S(t-),t-) = B(S(t-),t-,T)$, Equation (34) becomes

$$\begin{aligned}
&\left(\frac{\partial A}{\partial t}(S(t-),t-) + \frac{\partial B'}{\partial t}(S(t-),t-)X(t-)\right) \\
&+B'(S(t-),t-)\left(\widetilde{\Theta_0}(S(t-),t-) + \widetilde{\Theta_1}(S(t-),t-)X(t-)\right) \\
&+\frac{1}{2}\left(\Sigma_0'(S(t-),t-)B^2(S(t-),t-) + (B^2)'(S(t-),t-)\Sigma_1(S(t-),t-)X(t-)\right) \\
&+\int_U \left(e^{\Delta_S A(S(t-),t-) + \Delta_S B'(S(t-),t-)X(t-)} - 1\right) e^{\widetilde{h_0}(u,S(t-),t-) + \widetilde{h}_1'(u,S(t-),t-)X(t-)} \eta(du) \\
&+E^Q_{J,t-}\left(e^{B'(S(t-),t-)J(S(t-),t-)} - 1\right) = \psi_0(S_t,t) + \psi_1(S_t,t)' X_t.
\end{aligned}$$

To solve for $A(S(t-),t)$ and $B(S(t-),t-)$, we apply the following log-linear approximations

$$e^{\left(\Delta_S B'(S(t-),t-) + \tilde{h}_1'(u,S(t-),t-)X_{t-}\right)} \approx 1 + \left(\Delta_S B'(S(t-),t-) + \tilde{h}_1'(u,S(t-),t-)X(t-)\right)$$

and

$$e^{\tilde{h}_1'(u,S(t-),t-)X(t-)} \approx 1 + \tilde{h}_1'(u,S(t-),t-)X(t-)$$

Under these approximations:

$$\begin{aligned}
&\left(e^{\Delta_S A(S(t-),t-) + \Delta_S B'(S(t-),t-)X(t-)} - 1\right) e^{\tilde{h}_0(u,S(t-),t-) + \tilde{h}_1'(u,S(t-),t-)X(t-)} \\
&= e^{\left(\Delta_S A(S(t-),t-) + \tilde{h}_0(u,S(t-),t-)\right) + \left(\Delta_S B'(S(t-),t-) + \tilde{h}_1'(u,S(t-),t-)X(t-)\right)} - e^{\tilde{h}_0(u,S(t-),t-) + \tilde{h}_1'(u,S(t-),t-)X(t-)} \\
&\approx e^{\Delta_S A(S(t-),t-) + \tilde{h}_0(u,S(t-),t-)}\left(1 + \Delta_S B'(S(t-),t-) + \tilde{h}_1'(u,S(t-),t-)X(t-)\right) \\
&\quad - e^{\tilde{h}_0(u,S(t-),t-)}\left(1 + \tilde{h}_1'(u,S(t-),t-)X(t-)\right)
\end{aligned}$$

$$= e^{\tilde{h}_0(u,S(t-),t-)}\left(e^{\Delta_S A(u,S(t-),t-)}-1\right)+e^{\tilde{h}_0(u,S(t-),t-)}$$
$$\left[e^{\Delta_S A(u,S(t-),t-)}\left(\Delta_S B'\left(u,S\left(t-\right),t-\right)+\tilde{h}_1'\left(u,S\left(t-\right),t-\right)\right)-\tilde{h}_1'\left(u,S\left(t-\right),t-\right)\right]X\left(t-\right)$$

Now by matching coefficients, we obtain a set of differential equations for $A(S,t-)=A\left(S\left(t-\right),t-,T\right)$ and $B(S,t-)=B\left(S\left(t-\right),t-,T\right)$ which can be solved for the term structure of interest rates.

Proposition 4.2 *The price at time t- of a default-free pure discount bond with maturity T-t is given by* $P\left(t-,T\right)=e^{A(S,t-)+B(S,t-)'X_{t-}}$ *and the T- t-period interest rate is given by* $R\left(t-,T\right)= -\frac{A(S,t-)}{T-t}-\frac{B(S,t-)'X_{t-}}{T-t}$, *where* $A(S,t-)$ *and* $B(S,t-)$ *are determined by the following differential equations*

$$-\frac{\partial B(S,t-)}{\partial t}+\widetilde{\Theta}_1\left(S,t\right)'B\left(S,t-\right)+\frac{1}{2}\Sigma_1\left(S,t-\right)B^2\left(S,t-\right)$$
$$+\int_U\left[e^{\Delta_S A}\left(\Delta_S B+\tilde{h}_1\left(u;S,t-\right)\right)-\tilde{h}_1\left(u;S,t-\right)\right]e^{\tilde{h}_0(u;S,t-)}\eta\left(du\right)=\psi_1\left(S\right) \qquad (35)$$

and

$$-\frac{\partial A(S,t-)}{\partial t}+\widetilde{\Theta}_0\left(S\right)'B\left(S,t-\right)+\frac{1}{2}B^2\left(S,t-\right)'\Sigma_0\left(S,t-\right)$$
$$+\tilde{\delta}_J\left(S,t-\right)'E_J\left[e^{J'B(S,t-)}-1\right]+\int_U\left[e^{\Delta_S A}-1\right]e^{\tilde{h}_0(u;S,t-)}\eta\left(du\right)=\psi_0\left(S\right) \qquad (36)$$

with boundary conditions $A\left(S,0\right)=0$ *and* $B\left(S,0\right)=0$, *where* $\Delta_S A=A(u,t)-A(S,t-)$, $\Delta_S B=B(u,t)-B(S,t-)$, and $B^2\left(S,t\right)=\left(B_1^2\left(S,t\right),...,B_L^2\left(S,t\right)\right)'$.

5 Conclusions

Interest rate is probably one of the more important macroeconomic variables that are closely watched by financial market participants and policy makers. A great deal of research efforts have been devoted to econometric modeling of the dynamic behavior of interest rates. This chapter provides a brief review of models of the term structure of interest rates under regime shifts. These models retain the same tractability of the single-regime affine models on one hand, and allow more general and flexible specifications of the market prices of risk, the dynamics of state variables as well as the short-term interest rates on the other hand. These additional flexibilities can be important in capturing some silent features of the term structure of interest rates in empirical applications. The models can also be applied to analyze the implications of regime-switching for bond portfolio allocations and the pricing of interest rate derivatives.

An implicit assumption in the models reviewed in this chapter is that investors can observe the regimes. One extension is to assume that regimes are not observable or can only be imperfectly observed by investors. For example, if regimes represent different stance of the monetary policy, investors have to use observable state variables such as interest rates to learn the current regime when the monetary policy is not completely transparent.

Another possible extension is to tie the regimes more closely to economic fundamentals. Most regime-switching models are motivated by the business cycle fluctuations or shifts in policy regimes. In the term structure models reviewed in this chapter, however, regimes are only identified by the dynamics of interest rates and lack clear economic interpretations. Introducing other macroeconomic variables that are better indicators of the business cycle or the policy as additional state variables can give us deeper insight into the dynamic properties of interest rates across different economic regimes.

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