Statistical Analysis of the Filtering Model for Financial Ultra-High Frequency (UHF) Data

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Outline

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UHF Data and Marked Poisson Process

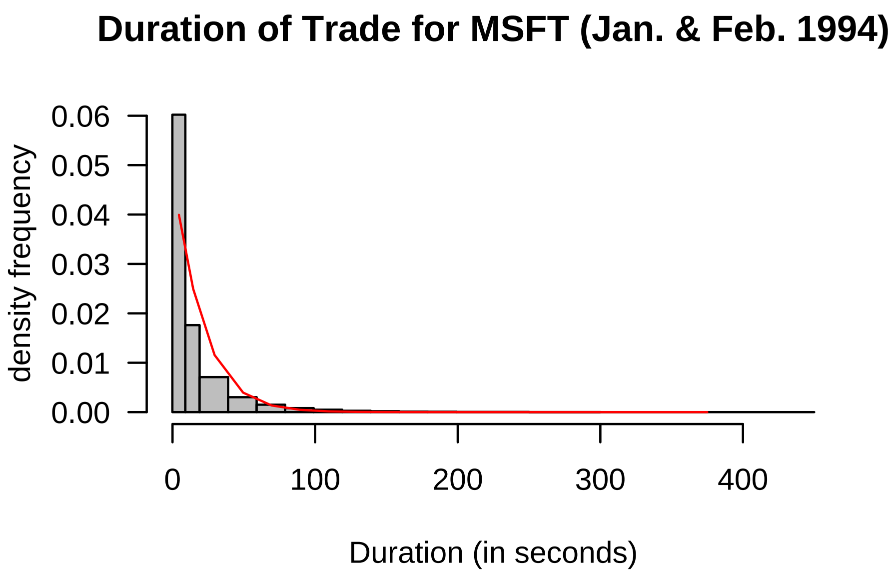
Two Characteristics of UHF Data

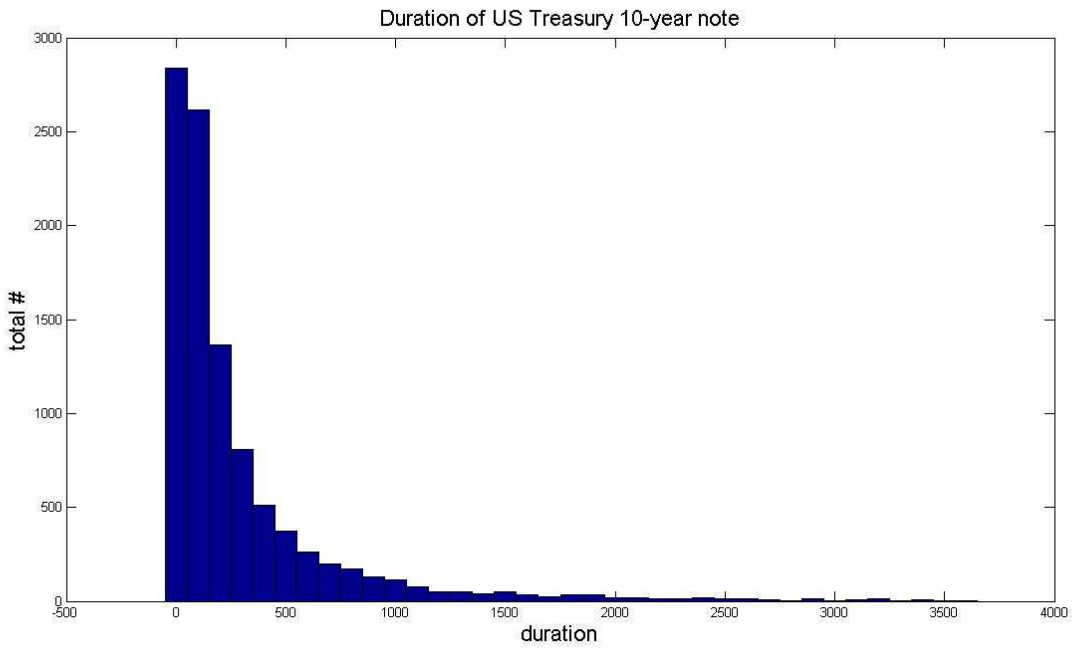
- Observations occur at varying random time intervals
- Market microstructure noises (frictions) are in price data

Marked Point Process and Marked Poisson Process

- Marked Point Process: $\{(T_n, X_n)\}$, where $\{T_n\}$ increasing, i.e. $T_n \leq T_{n+1}$.
- Marked Poisson Process: $\{(T_n, X_n)\}$ with $\{T_n\}$ from a conditional Poisson Process (or Cox process, or doubly stochastic Poisson).

Durations or Inter-trading Times





An Irregularly-Spaced Time Series

- Engle (2000) – Data: $\{(\Delta t_i, y_i), i = 1, ..., N\}$ where $\Delta t_i = t_i - t_{i-1}$.
- **Conditional density:**

$$(\Delta t_i, y_i)|\mathcal{F}_{i-1} \sim f(\Delta t_i, y_i|\breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta)$$

where $\breve{z}_i = \{z_i, z_{i-1}, ..., z_1\}$.

$$f(\Delta t_i, y_i|\breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta) = g(\Delta t_i|\breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta) q(y_i|\Delta t_i, \breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta) \quad (1)$$

- **Log likelihood:**

$$l(\Delta, Y; \theta) = \sum_{i=1}^{N} \log g(\Delta t_i|\breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta) + \sum_{i=1}^{N} \log q(y_i|\Delta t_i, \breve{\Delta} t_{i-1}, \breve{y}_{i-1}; \theta)$$

Related Developments

- Autoregressive Conditional Duration (ACD) model by Engle and Russell (1998), and logarithmic ACD by Bauwens and Giot (2000), threshold ACD by Zhang, Russell and Tsay (2001), Asymmetric ACD by Bauwens and Giot (2003)
- Ordered Probit model by Hausman, Lo and Mackinlay (1992), Price Change Duration model by McCulloch and Tsay (2001), Activity -Direction -Size model by Rydberg and Shepherd (2003), Autoregressive Conditional Multinomial- ACD by Russell and Engle (2005)
- Bivariate Point process model by Engle and Lunde (2003), and Autoregressive Conditional Intensity model by Russell (1999)
- UHF-GARCH by Engle (2000), ACD-GARCH by Ghysels and Jasiak (1998), Engle and Zheng (2007)

- **The Second View:**
A Realized Sample Path of MPP: Zeng (2003)

Construction of Price from Value

Three-Step Construction:

- *Intrinsic value process:* $X(t)$

Assumption 1.1: Markov process, (θ, X), is the solution of a martingale problem for a generator $\mathbf{A}$ such that

$$M_f(t) = f(\theta(t), X(t)) - \int_0^t \mathbf{A} f(\theta(s), X(s)) ds$$

is a $\mathcal{F}_t^{\theta,X}$-martingale.

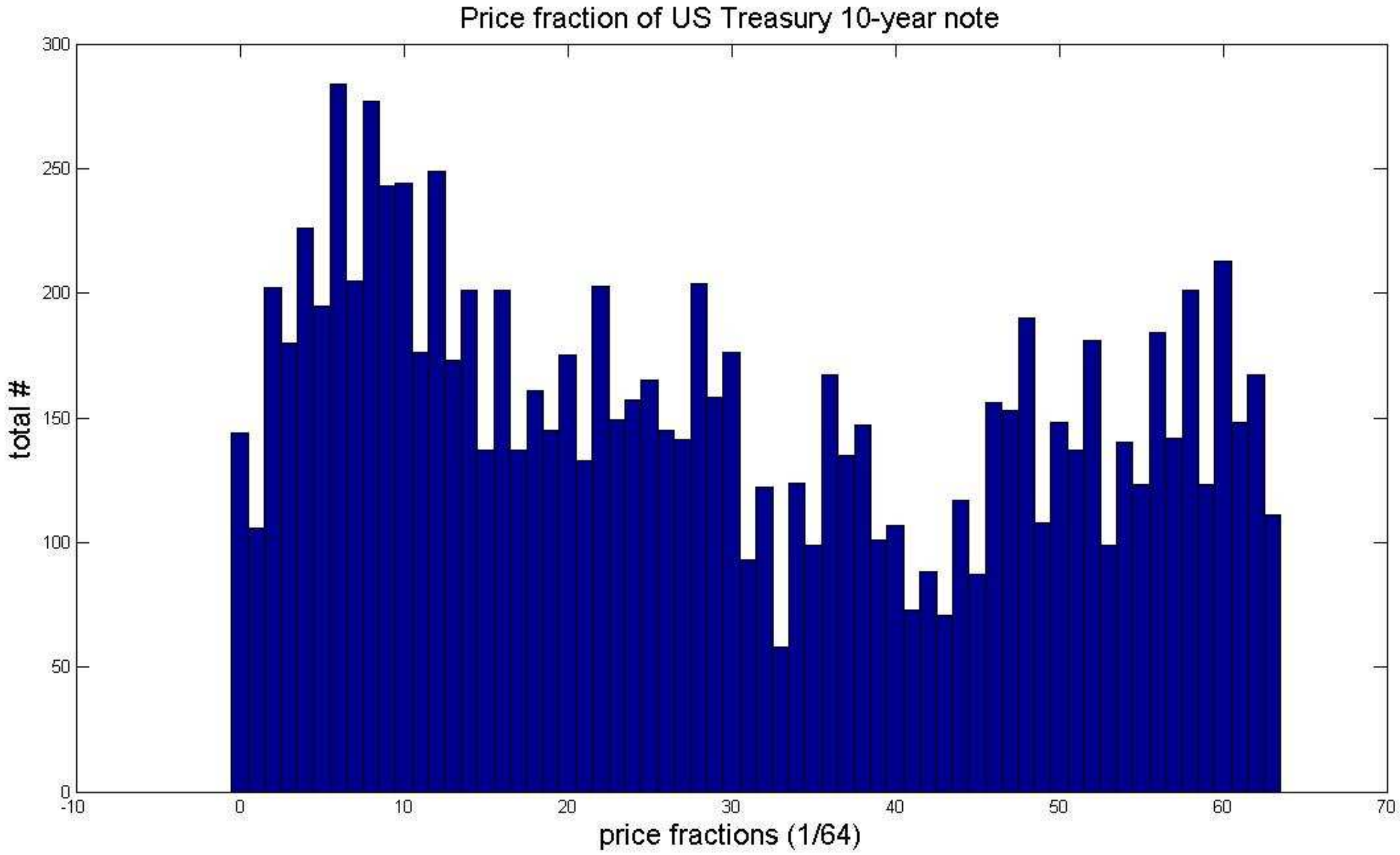
- *Trading times:* $t_1, t_2, \ldots, t_i, \ldots$ follows a *conditional Poisson process* with $a(\theta(t), X(t), t)$.

- *Price* at t_i: $Z(t_i) = F(X(t_i)) = b_i\Big(R[X(t_i) + V_i, \frac{1}{M}]\Big)$

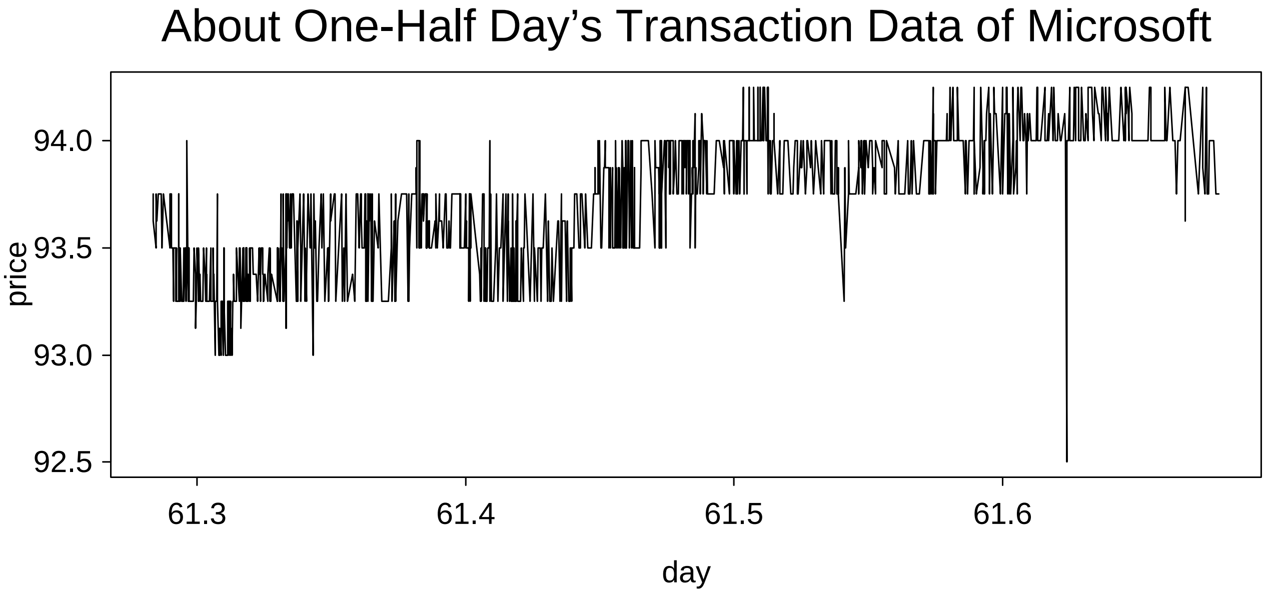
where F, a random transformation modeling market microstructure noise, is specified by the transition probability $p(Z(t_i)|X(t_i);\theta)$.

Remark: Closely related to (1) the structure models in Hasbrouck (1996) and (2) the TS Models for RV in ZMA (2005), Bandi and Russell (2006), Fan and Wang (2007), Li and Mykland (2007) and others.

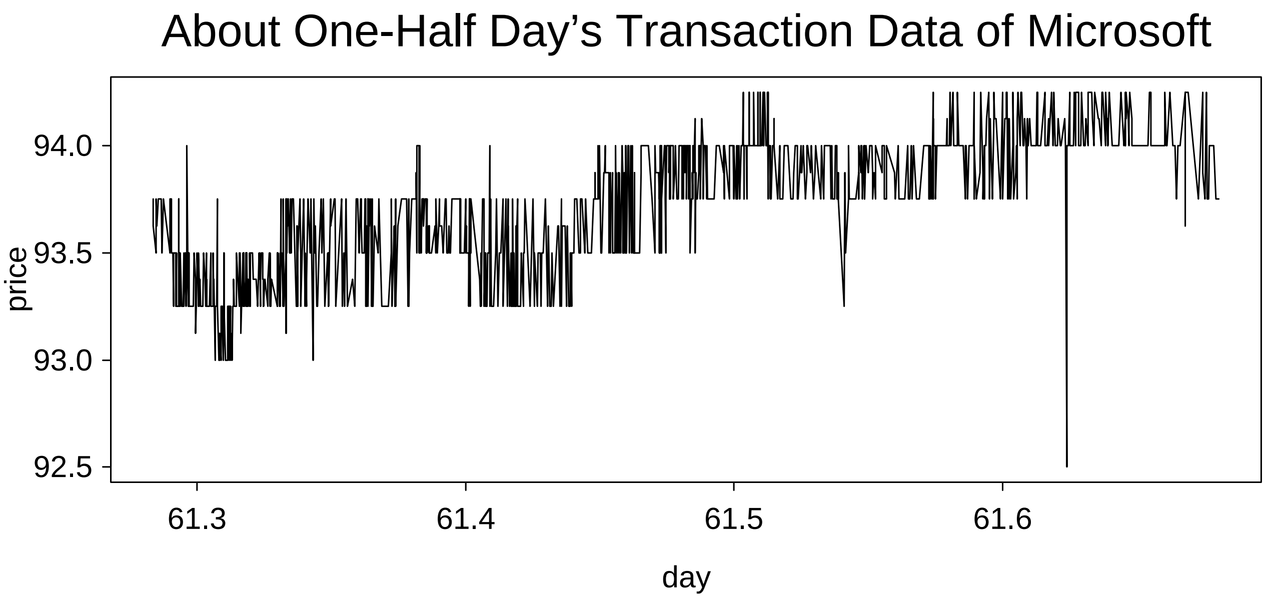
Price Clustering in Treasury Notes



A Collection of Counting Processes



A Collection of Counting Processes



$$\vec{Y}(t) = \begin{pmatrix} N_1(\int_0^t \lambda_1(\theta(s), X(s), s)ds) \\ N_2(\int_0^t \lambda_2(\theta(s), X(s), s)ds) \\ \vdots \\ N_n(\int_0^t \lambda_n(\theta(s), X(s), s)ds) \end{pmatrix}, \quad (3)$$

where $Y_j(t) = N_j(\int_0^t \lambda_j(\theta(s), X(s), s)ds)$ records the cumulative # of trades that have occurred at the jth price level up to time t.

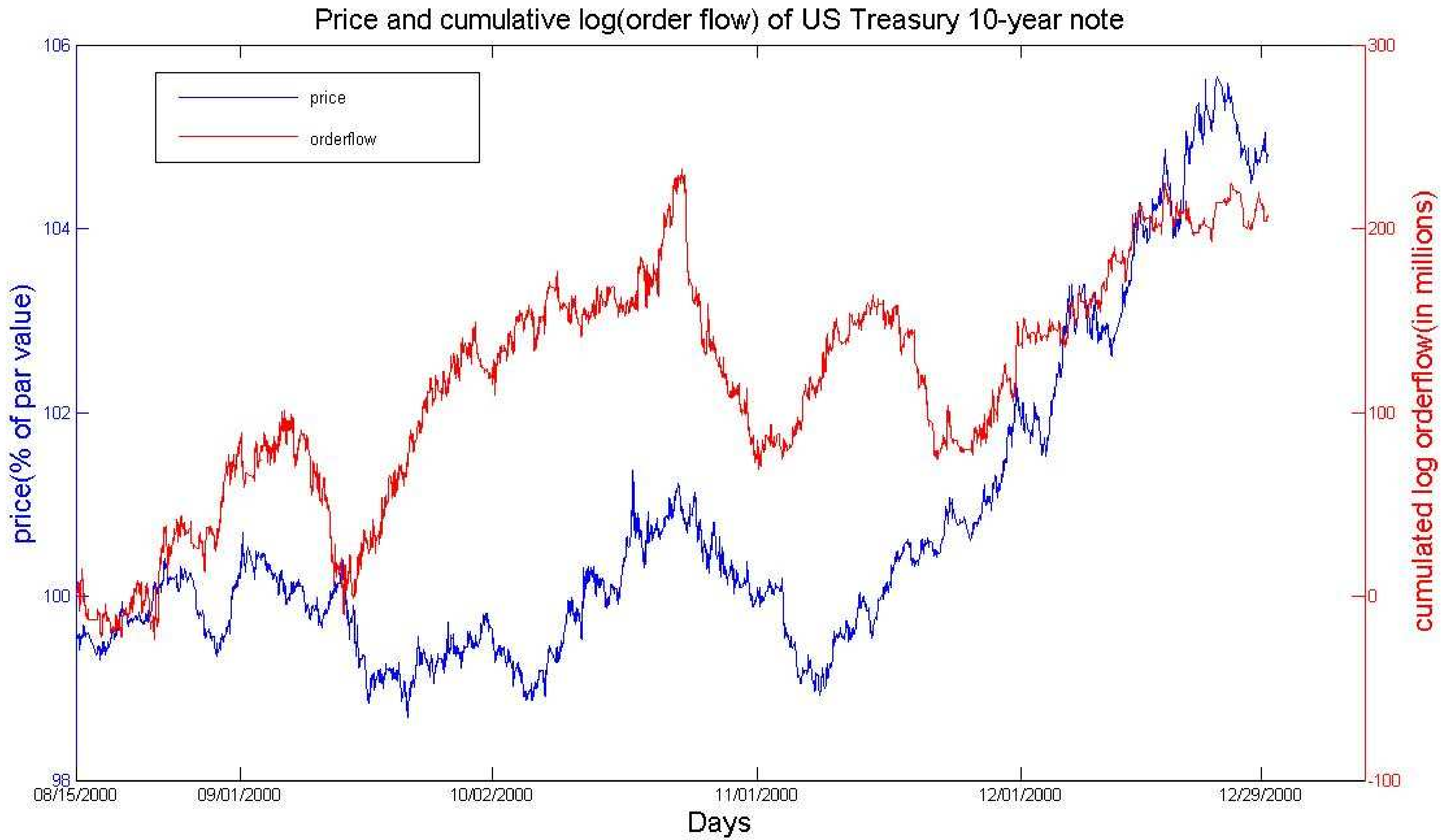
Other Assumptions of Model I

Filtering with counting process observations

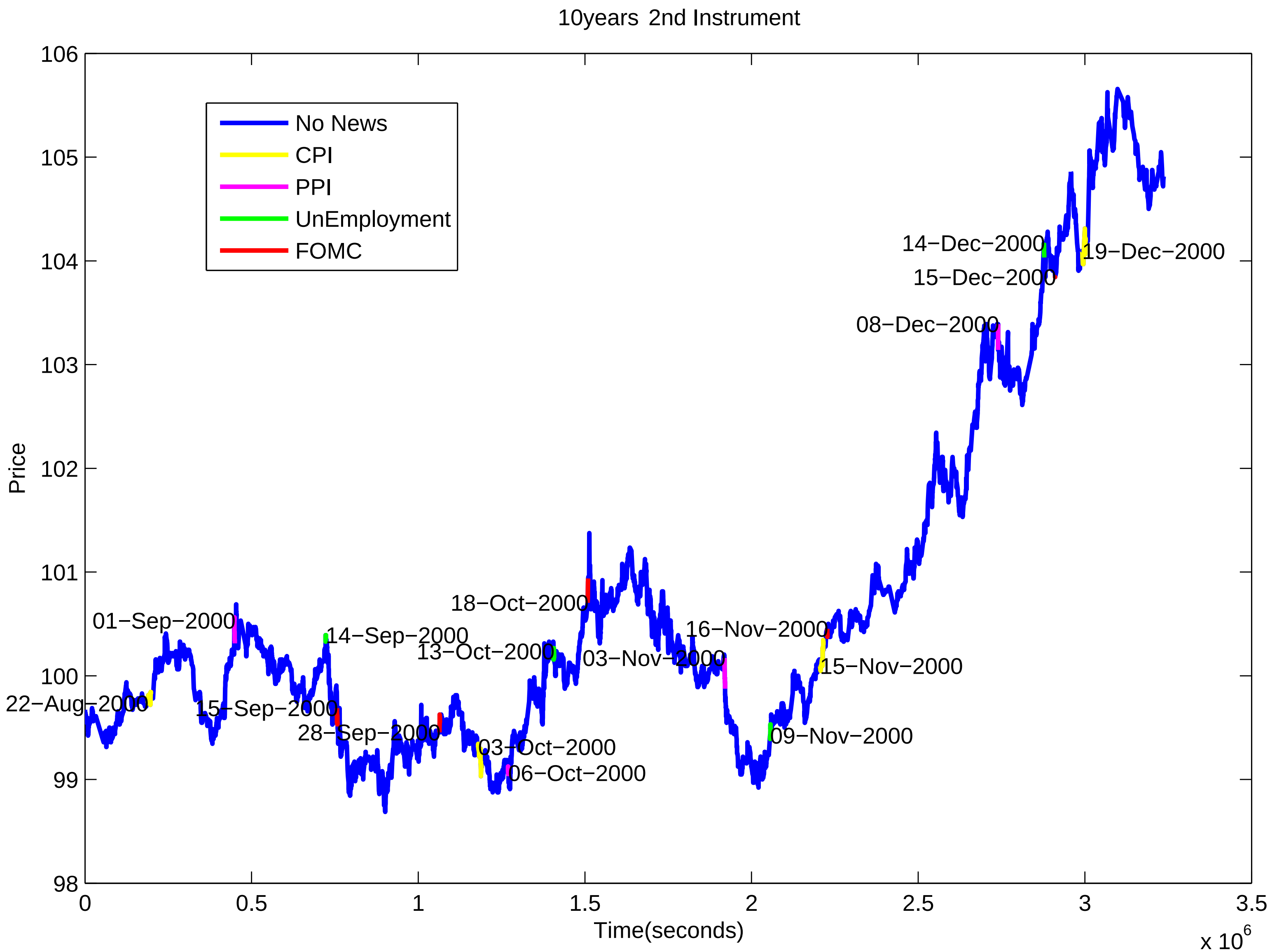
- **Assumption 1.2:** $(N_1, \ldots, N_n)$ are unit Poisson processes under measure P.
- **Assumption 1.3:** $(\theta, X), N_1, \ldots, N_n$ are independent under P.
- **Assumption 1.4:** $0 \leq a(\theta(t), X(t), t) \leq C$ for some $C > 0$ and all $\theta(t), X(t), t > 0$.
- **Assumption 1.5:** Intensities: $\lambda_j(\theta, x, t) = a(\theta, x, t)p(y_j|x)$, where $a(x, \theta, t)$ is the total trading intensity, and $p_j = p(y_j|x)$ is the transition probability from x to y_j.

Signal: (θ, X) **Observation:** $\vec{Y}$ or Z.

Price and Order Flow



Price and Economic Announcements



A Motivating Example

V_t: observable variable such as *economic news, signed order flow, signed trade and others* in Treasury market.

- **Intrinsic value process:**

$$\frac{dX_t}{X_t} = (\mu + \kappa_\mu V_t)dt + (\sigma + \kappa_\sigma V_t)dB_t$$

- **Trading intensity:** $a(V^t, Z^t, t)$ is *non-anticipating*, where $V^t(\cdot) = V(\cdot \wedge t)$. E.g. $a(V(t_{i-1}), Z(t_{i-1}), \Delta_{i-1}, t)$ for $t_{i-1} \leq t < t_i$. This allows *Exponential ACD Model*.
- **Price** at time t_i:

$$Z(t_i) = F(X(t_i))$$

where F is a random transformation specified by a *non-anticipating* transition probability $p(Z(t_i)|X(t_i); \theta^{t_i-}, X^{t_i-}, Z^{t_i-}, V^{t_i-},)$.
E.g. $p(Z(t_i)|X(t_i); \theta, Z(t_{i-1}) - X(t_{i-1}))$.
This further allows *serial dependent microstructure noise*.

Random-arrival-time State-Space Model

Three-Step Construction:

- **State process:** $X(t)$.

Assumption 2.1: is the same as Assumption 1.1, but both θ and X can be vector processes, and $M_f(t)$ is a $\mathcal{F}_t^{\theta,X,V}$-martingale.

- **Event times:** $t_1, t_2, \ldots, t_i, \ldots$ follows a conditional Poisson process with a non-anticipating intensity $a(\theta^t, X^t, V^t, Z^t, t)$.
- **Observation** at t_i:

$$Z(t_i) = F(X(t_i))$$

where F is a random transformation with the non-anticipating transition probability $p(Z(t_i)|X(t_i); \theta^{t_i-}, X^{t_i-}, V^{t_i-}, Z^{t_i-}, t))$.

Filtering with MPP Observations

- **Setup:** Mark space: U; measure space: $(U, \mathcal{U}, \mu)$, μ: finite measure; ξ is a Poisson Random Measure (PRM) on $\mathcal{U} \times \mathcal{B}[0,\infty) \times \mathcal{B}[0,\infty)$ with mean measure $\mu \times m \times m$. For $A \in \mathcal{U}$,

$$Y(A,t) = \int_{A\times[0,t]\times[0,+\infty)} \mathbf{I}_{[0,\lambda(\theta^s,X^s,V^s,Z^s;u,s)]}(v)\xi(du \times ds \times dv),$$

where $Y(A,t)$ is a counting process recording the cumulative number of events that have occurred in the set A up to time t.

$$\tilde{Y}(A,t) = Y(A,t) - \int_{A\times[0,t]} \lambda(\theta^s, X^s, V^s, Z^s; u, s)\mu(du)ds$$

is a martingale.

Signal: (θ, X) **Observation:** (Y, V) or (Z, V).

The Reference Measure

- Under the reference measure $\mathbf{Q}$, For $A \in \mathcal{U}$,

$$Y(A,t) = \int_{A\times[0,t]\times[0,+\infty)} \mathbf{I}_{[0,1]}(v)\xi(du \times ds \times dv),$$

where $Y(A,t)$ is a counting process recording the cumulative number of events that have occurred in the set A up to time t.

$$\tilde{Y}(A,t) = Y(A,t) - \int_{A\times[0,t]} \mu(du)ds = Y(A,t) - \mu(A)t$$

is a martingale.

- **For Model I:** $U = \{1, 2, \cdots, n\}$, $A = j$ and V and Z are not in the intensity,
Under $\mathbf{P}$, $Y(A,t) = Y(j,t) = N_j(\int_0^t \lambda_j(\theta(s), X(s), s)ds)$.
Under $\mathbf{Q}$, $Y(A,t) = Y(j,t) = Y_j(t)$ is a unit Poisson process.

Other Assumptions of Model II

- **Assumption 2.2:** ξ is a PRM with the mean measure $\mu \times m \times m$ under P, where μ is finite measure.
- **Assumption 2.3:** (θ, X) and ξ are independent under measure P.
- **Assumption 2.4:** $0 \leq a(\theta^t, X^t, V^t, Z^t, t) \leq C$ for some $C > 0$ and all possible $\theta^t, X^t, V^t, Z^t, t$.
- **Assumption 2.5:** Stochastic intensity kernel (non-anticipating):

$$\lambda(\theta^{t-}, x^{t-}, v^{t-}, z^{t-}; u, t-) = a(\theta^{t-}, x^{t-}, v^{t-}, z^{t-}, t-)p(u|x; \theta^{t-}, x^{t-}, v^{t-}, z^{t-}) \tag{4}$$

where $p(u|x; \theta^{t-}, x^{t-}, v^{t-}, z^{t-}))$ is the transition probability from $X(t)$ to u.

Remark: Note the similarity between Eq.(4) and Eq.(1).

Examples I

Group I: without continuous-time latent X_t

- $U = \{1\}$: Cox process and Exponential ACD model (Engle and Russell 1998).
- $U = \{\frac{a}{M}, \frac{a+1}{M}, \cdots, \frac{b}{M}\}$, or R, or R^+: UHF-GARCH model and many previously reviewed models under the framework of Engle (2000).

Group II: with latent X_t not depending on V

- Zeng (2003) and its extension to multi-stocks (Scott and Zeng 2008).
- Estimating Volatility via filtering: Frey and Runggaldier (2001), Cvitanic, Liptser and Rozovskii (2006), Ceci and Gerardi (2007a,b).
- Estimating Markov process sampled at conditional Poisson time: Duffie and Glenn (2004).
- Classical examples: Segall, Davis and Kailath (1975), Bremaud (1981), Liptser and Shiryayev (1978), Kliemann, Koch and Marchetti (1990), and Last and Brandt (1995).

Examples II

Group III: with latent X_t depending on V

- V_t can change at random times (such as trading time) or deterministic times (such as every 5 minutes or every day).
- V_t can takes values such as ± 1 for the indicator of buyer or seller initiating trade; or 0 or 1 for the indicator of a special period; or other values such as order flow or a function of order flow.
- When X_t is GBM, V_t can be added in instantaneous expected return, or instantaneous volatility, or in the trading noise. Or find the “best" $f(V_t)$ for the model.
- X_t can be O-U process; CIR model; CES model; SV models; plus jumps; plus regime-switching; with spikes; α-stable process; and others. Then, V_t can be added in the related parameters with different economic interpretations.

- Z can be trading prices or ask and bid quotes.

An Integral Form of Price

- Let $Z(t)$ be the price of the most recent transaction at or before time t.

$$Z(t) = Z(0) + \int_{[0,t]\times U} (u - Z(s-))Y(du \times ds).$$

$\mathbf{Remarks:}$

- This is the telescoping sum: $Z(t) = Z(0) + \sum_{t_i \le t}(Z(t_i) - Z(t_{i-1})$. This form is similar to that of Skorohod (1989) for continuous time Markov chain.

- This form is essential for the *risk minimization hedging* (Lee and Zeng 2006, for Model I), and the *mean-variance portfolio selection* problem of the model (Xiong and Zeng 2007, for Model I).

Joint Likelihood Function

- Continuous-time joint likelihood function of (θ, X, Y):
- For Model I,

$$L(t) = \frac{dP}{dQ}(t) = \exp\Big\{ \sum_{k=1}^{n} \int_0^t \log \lambda_k(\theta(s-), X(s-), s-) dY_k(s)$$
$$- \sum_{k=1}^{n} \int_0^t \Big[\lambda_k(\theta(s-), X(s-), s) - 1 \Big] ds \Big\}.$$

- For Model II,

$$L(t) = \exp\Big\{ \int_0^t \int_U \log \lambda(\theta^{s-}, X^{s-}, V^{s-}, Z^{s-}; u, s-) Y(du \times ds)$$
$$- \int_0^t \int_U \Big[\lambda(\theta^{s-}, X^{s-}, V^{s-}, Z^{s-}; u, s) - 1 \Big] \mu(du) ds \Big\}$$

Likelihoods and Posterior

Define: $\phi(f,t) = E^Q[f(\theta(t), X(t))L(t)|\mathcal{F}_t^{Y,V}]$. Then, $\phi(1,t) = E^Q[L(t)|\mathcal{F}_t^{Y,V}]$ is the *likelihood* of Y *or* the *integrated (marginal) likelihood* of Y after assigning a prior to $(\theta(0), X(0))$.

Define: π_t is the conditional distribution of $(\theta(t), X(t))$ given $\mathcal{F}_t^{Y,V}$.
π_t becomes the *posterior* after a prior is assigned.
Define: $\pi(f,t) = E^P[f(\theta(t), X(t))|\mathcal{F}_t^{Y,V}] = \int f(\theta, x)\pi_t(d\theta, dx)$.

- Bayes theorem gives:

$$\pi(f,t) = \frac{\phi(f,t)}{\phi(1,t)}.$$

SPDE for ϕ_t – unnormalized filtering equation
SPDE for π_t – normalized filtering equation

Bayes Factor and Likelihood Ratio

Suppose there are two models: Model 1 and Model 2.

Define: two *conditional ratio processes*:

$$q_1(f_1, t) = \frac{\phi_1(f_1, t)}{\phi_2(1, t)} \quad \text{and} \quad q_2(f_2, t) = \frac{\phi_2(f_2, t)}{\phi_1(1, t)}$$

The *Bayes Factors:* (BF: the ratio of two integrated likelihoods)

$$BF_{12} = \frac{\phi_1(1, t)}{\phi_2(1, t)} = q_1(1, t) \quad \text{and} \quad BF_{21} = \frac{\phi_2(1, t)}{\phi_1(1, t)} = q_2(1, t)$$

- *Strongly Reject Model 1* if BF_{21} is larger than 20.
- *Decisively Reject Model 1* if BF_{21} is larger than 150.

Unnormalized Filtering Equation

- **Theorem 1:** (Zeng 2003, for Model I) *Under Assumptions 2.1–2.5, ϕ_t is the unique measure-valued solution of the following SPDE, called the unnormalized filtering equation:*

$$\phi(f,t) = \phi(f,0) + \int_0^t \phi(\mathbf{A}f, s)ds - \int_0^t \int_U \phi(f(\lambda(u)-1), s)\mu(du)ds$$

$$+ \int_0^t \int_U \phi(f(\lambda(u)-1), s-)Y(du \times ds),$$

for every $t > 0$ and $f \in D(\mathbf{A})$.

Normalized Filtering Equation

Theorem 1: (continued) π_t *is the unique measure-valued solution of the following SPDE, called the normalized filtering equation:*

$$\pi(f,t) = \pi(f,0) + \int_0^t \pi(\mathbf{A}f,s)ds + \int_0^t \pi(f,s)\int_U \pi(\lambda(u),s)\mu(du)ds$$

$$-\int_0^t \int_U \pi(f\lambda(u),s)\mu(du)ds + \int_0^t \int_U \Big[\frac{\pi(f\lambda(u),s-)}{\pi(\lambda(u),s-)} - \pi(f,s-)\Big] dY(du \times ds)$$

Remark: When $a(\theta^t, X^t, V^t, Z^t, t) = a(V^t, Z^t, t)$ (including the case of *EACD model*), it can be simplified as:

$$\pi(f,t) = \pi(f,0) + \int_0^t \pi(\mathbf{A}f,s)ds + \int_0^t \int_U \Big[\frac{\pi(fp(u),s-)}{\pi(p(u),s-)} - \pi(f,s-)\Big] dY(du \times ds)$$

where $p(u) = p(u|X(t);\theta^t, X^t, V^t, Z^t, t)$ allowing *dependent noise*.

Evolution Equations for BF

- **Theorem 2:** (Kouritzin and Zeng 2005, for Model I) *Assume Model 1 has* $(\mathbf{A_1}, \lambda_1, \mu_1)$ *and Model 2 has* $(\mathbf{A_2}, \lambda_2, \mu_2)$*. Both models satisfy Assumptions 2.1–2.5. Then,* $(q_{1,t}, q_{2,t})$ *is the unique pair measure-valued solution of the following system of SPDEs,*

$$q_1(f_1, t) = q_1(f_1, 0) + \int_0^t q_1(\mathbf{A_1} f_1, s) ds$$

$$+ \int_0^t \frac{q_1(f_1, s)}{q_2(1, s)} \int_U q_2(\lambda_2(u), s)\mu_2(du)ds - \int_0^t \int_U q_1(f_1\lambda_1(u), s)\mu_1(du)ds$$

$$+ \int_0^t \int_U \Big[\frac{q_1(f_1\lambda_1(u), s-)}{q_2(\lambda_2(u), s-)} q_2(1, s-) - q_1(f_1, s-)\Big] dY(du \times ds)$$

and

$$q_2(f_2, t) = ...$$

A Consistency Theorem

- **Theorem 3:** (Zeng 2003, and Kouritzin and Zeng 2005, for Model I)

Suppose that Assumptions 2.1 to 2.5 hold for (θ, X, Y) *and* $(\theta_\epsilon, X_\epsilon, Y_\epsilon)$*.*
If $(\theta_\epsilon, X_\epsilon) \Rightarrow (\theta, X)$ *as* $\epsilon \to 0$*, then for bounded continuous functions,* f*,*
(i) $Y_\epsilon \Rightarrow Y$*,* *(ii)* $\phi_\epsilon(f,t) \Rightarrow \phi(f,t)$*,* *(iii)* $\pi_\epsilon(f,t) \Rightarrow \pi(f,t)$*.*
In the two-model case for model selection, then
(iv) $q_{k,\epsilon}(f_k,t) \Rightarrow q_k(f_k,t)$ *for* $k = 1, 2$ *simultaneously.*

Sketch of Proof:

- First, use Kurtz and Protter (1991)'s theorem on convergence of stochastic integral and the Continuous Mapping theorem to prove $L_\epsilon \Rightarrow L$. Then, $((\theta_\epsilon, X_\epsilon), Y_\epsilon, L_\epsilon) \Rightarrow ((\theta, X), Y, L)$.
- Second, use Goggin (1994)'s or Kouritzin and Zeng (2005)'s theorems on convergence of conditional expectations and the Continuous Mapping theorem to prove (ii), (iii) and (iv).

Markov Chain Approximation Method

Construction of Parallel Recursive Online Algorithms –
for computing nearly posterior, integrated likelihood and Bayes factors

For Example, to compute the *nearly* posterior: **Three Steps:**

- Construct a Markov chain $(\theta_\epsilon, X_\epsilon)$ to approximate (θ, X).
- Derive the filtering (or evolution) equations for $(\theta_\epsilon, X_\epsilon, Y_\epsilon)$.
  - *Propagation Equation*:

$$\pi_\varepsilon(f, t_{i+1}-) = \pi_\varepsilon(f, t_i) + \int_{t_i}^{t_{i+1}-} \pi_\varepsilon(\mathbf{A}_\varepsilon f, s)ds.$$

  - *Updating Equation*:

$$\pi_\varepsilon(f, t_{i+1}) = \frac{\pi_\varepsilon(fp(u), t_{i+1}-)}{\pi_\varepsilon(p(u), t_{i+1}-)}$$

- Convert the equation for $(\theta_\epsilon, X_\epsilon, Y_\epsilon)$ to recursive algorithms by
  - (a) representing $\pi_\epsilon(\cdot, t)$, for example, as a finite array with components being $\pi_\epsilon(f, t)$ for lattice-point indicator f;
  - (b) approximating the time integral with an Euler scheme.

Two FM Models for Stock Price

- Value Processes of the two Filtering Micromovement (FM) Models

1. GBM: (Zeng 2003)

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t,$$

$$\mathbf{A}f(x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} f(x) + \mu x \frac{\partial}{\partial x} f(x).$$

2. JSV-GBM: (Zeng 2004)

$$\frac{dX_t}{X_t} = \mu dt + \sigma(t) dW_t,$$

$$d\sigma(t) = (U_{N(t)} - \sigma(t-))dN(t)$$

where $N(t)$ is a Poisson process with intensity λ_σ and the jump size, $\{U_i\}$, are i.i.d random variables with uniform distribution on $[\alpha_\sigma, \beta_\sigma]$.

Noise for the Models

$$Y(t_i) = F(X(t_i)) = b_i(R[X(t_i), \frac{1}{M}] + D_i)$$

- **Discrete noise:** $R[x, \frac{1}{M}]$, rounding function, where $M = 8, 64, 128$.
- **Non-clustering noise:** $\{D_i\}$**,** has a doubly-geometric distribution:

$$P\{D = d\} = \begin{cases} 1 - (\rho + \kappa_\rho v) & \text{if } d = 0 \\ \frac{1}{2}[1 - (\rho + \kappa_\rho v)](\rho + \kappa_\rho v)^{64|d|} & \text{if } d = \pm\frac{i}{M} \text{ for } i = 1, 2, 3.... \end{cases}.$$

- **Clustering noise:** $b_i(\cdot)$**,** a random biasing function

biasing rule: Set $y' = R[X(t_i), \frac{1}{M}] + D_i$ and $y = Y(t_i) = b(y')$.

· If fractional part of y' is an even Mth, y stays on y' w. p. 1.

· If the fractional part of y' is an odd Mth, then

y' moves to the closest $M/2$th w.p. α,

or y' moves to the closest odd $M/4$th or integer w.p. β,

or y stays on y' w.p. $1 - \alpha - \beta$.

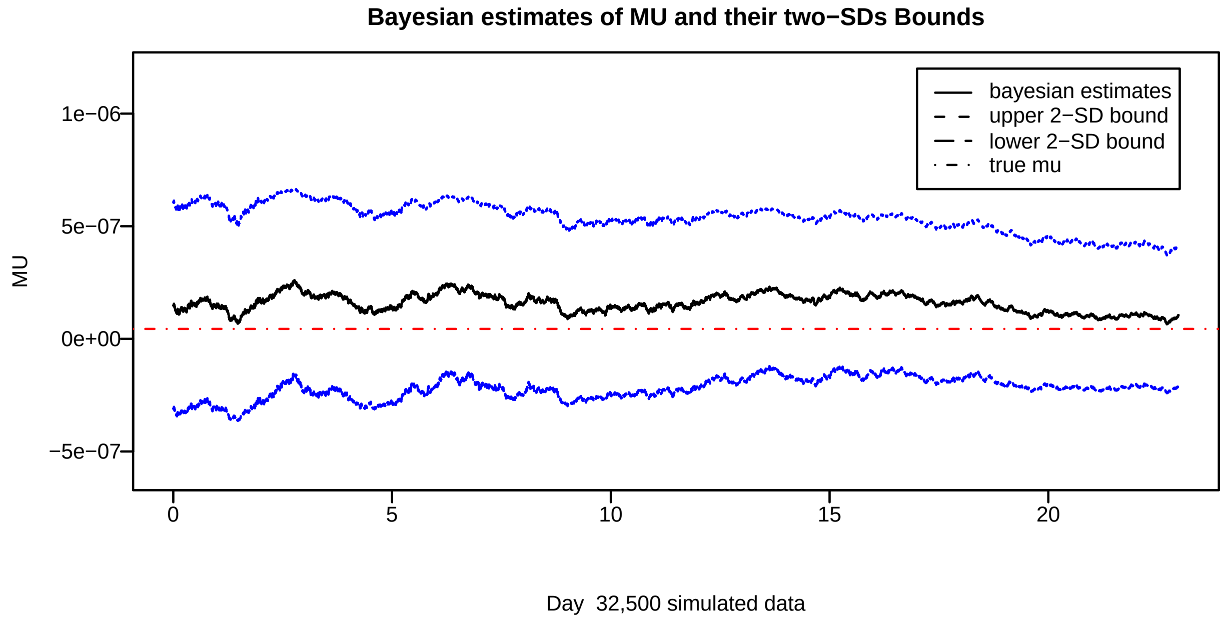
- Parameters in the Model : $(\mu, \sigma, \rho, \kappa_\mu, \kappa_\sigma, \kappa_\rho, \alpha, \beta)$.

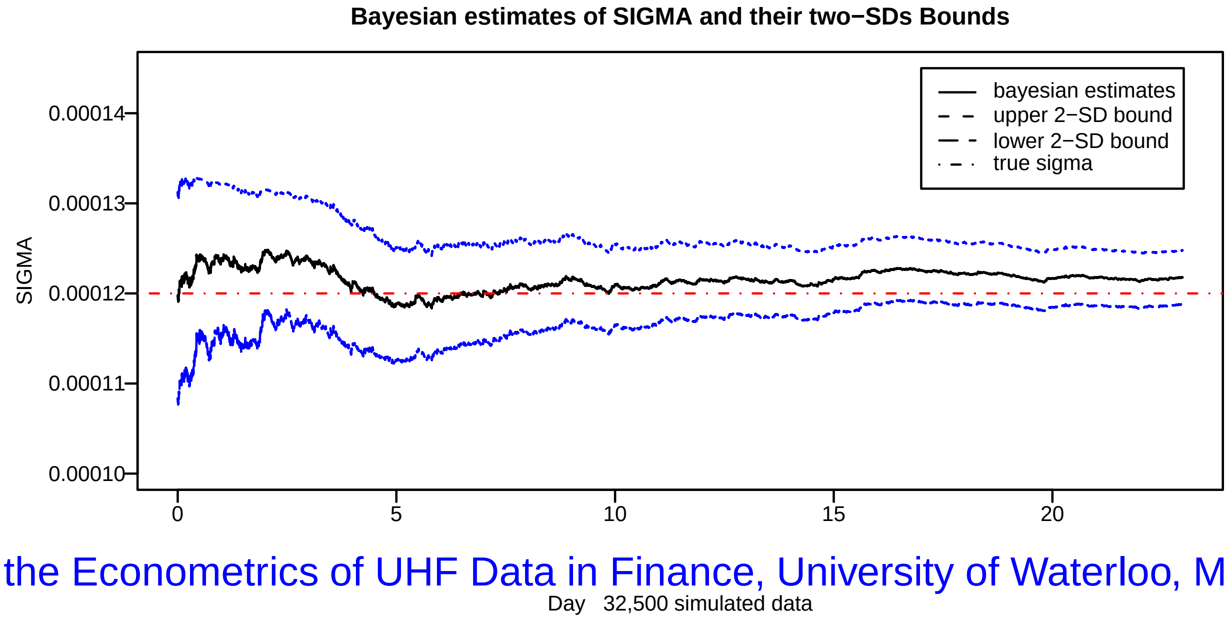
Consistency of Bayes Estimates (I)

- **Theorem 4:** (Zeng 2003) *For the simple filtering model (I) of GBM, assume that the clustering parameters* (α, β) *are known, and* (μ, σ, ρ) *has a prior. Then, the Bayes estimates are consistent almost surely. Namely,* $E[f(\mu, \sigma, \rho)|\mathcal{F}_t^Z] \to f(\mu, \sigma, \rho)$ *as* $t \to \infty$ *for bounded continuous function* f *when* $\mu - \frac{1}{2}\sigma^2 > 0$*.*

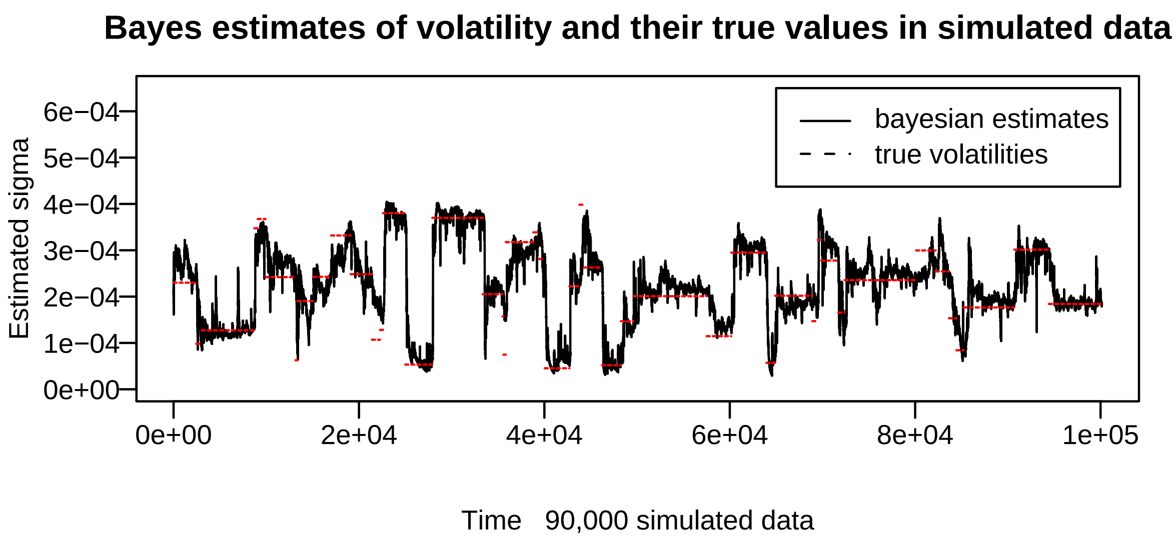
Remark: The standard condition "$\mu - \frac{1}{2}\sigma^2 > 0$" rules out bankruptcy.

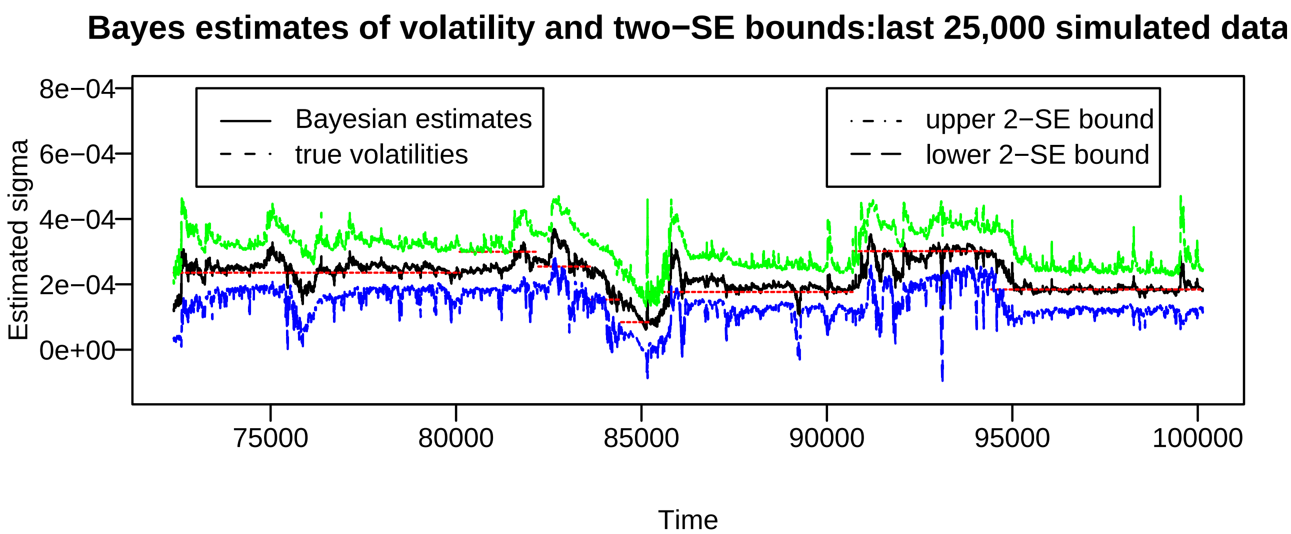
Bayes Estimates: Simulated Data I



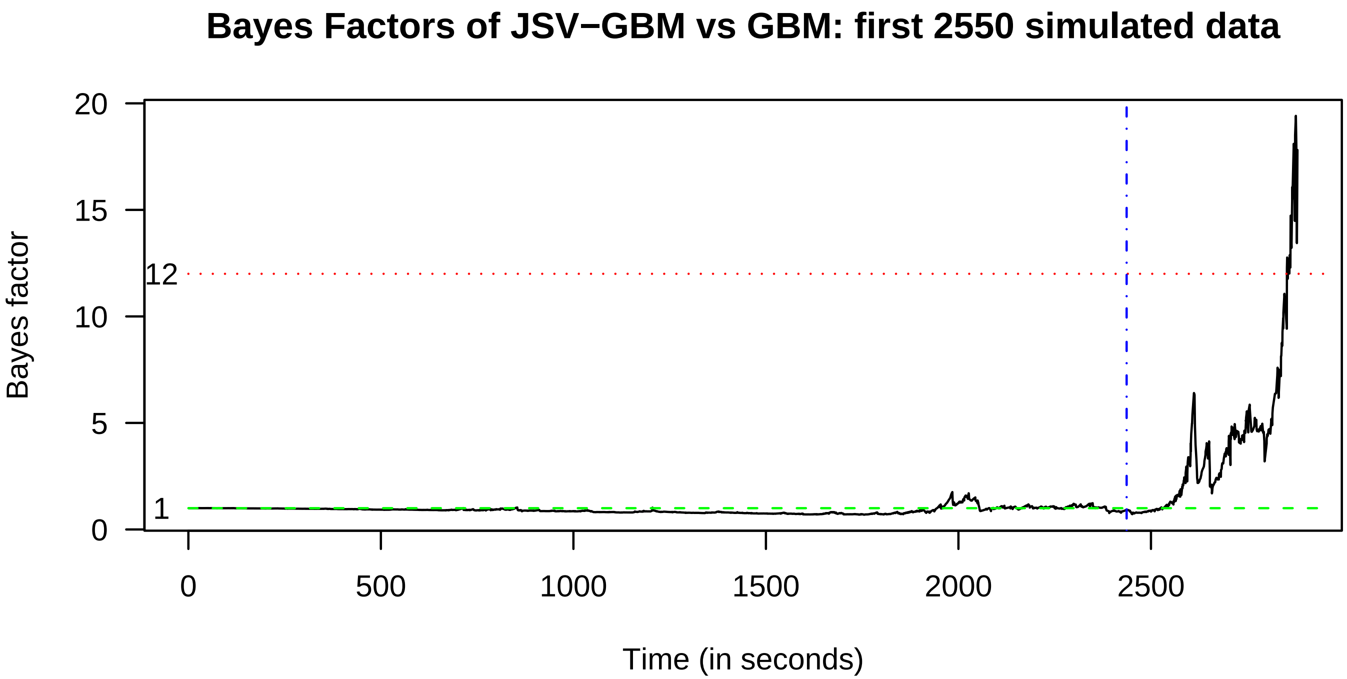


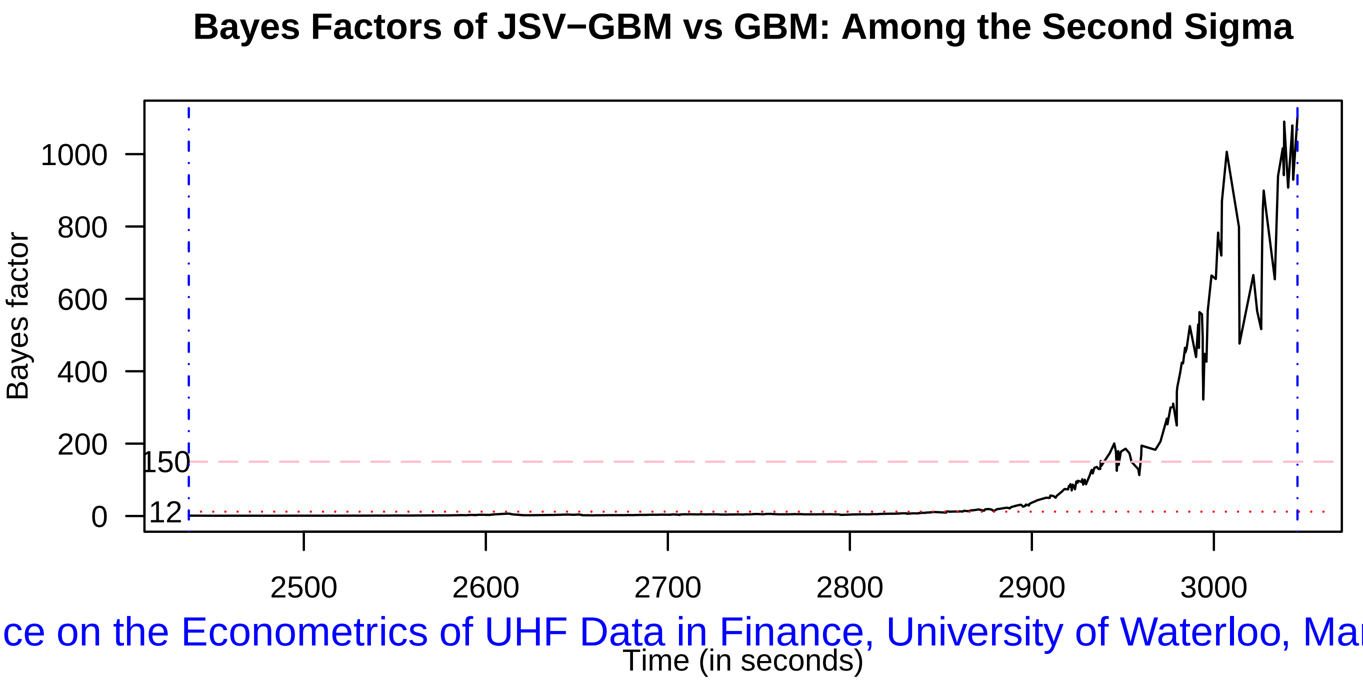
Bayes Estimates II: Simulated Data





Bayes Factor: Simulated Data





One FM Model for Treasury UHF Data

(Joint with D. Kuipers (UMKC, Finance) and X. Hu (Princeton, Econ))

- **Intrinsic value process:**

$$\frac{dX_t}{X_t} = \mu dt + (\sigma + \kappa_\sigma V_t) dB_t$$

Generator: $\theta = (\mu, \sigma, \kappa_\sigma, \rho, \kappa_\rho)$

$$\mathbf{A}_v f(x, \theta) = \mu x \frac{\partial}{\partial x} f(\theta, x) + \frac{1}{2} (\sigma + \kappa_\sigma v)^2 x^2 \frac{\partial^2}{\partial x^2} f(\theta, x).$$

where (1) V_t is the indicator of buyer (+1) or seller (-1) initiating;
(2) V_t is signed sqrt(Vol) with sign +(-) when buyer(seller) initiating;
(3) V_t is 1 when t is in a a specific time interval, otherwise 0.

- **Trading intensity:** $a(V^t, Z^t, t)$ independent of X_t can be modeled by an Exponential ACD Model.

Consistency of Bayes Estimates (II)

- **Theorem 5:** *For the above simple filtering model (III) of GBM with regression factors, assume that* $V_t = \pm 1$ *has infinite non-alternating* ± 1 *and the clustering parameters* (α, β) *are known, and* $\theta = (\mu, \sigma, \rho, \kappa_\mu, \kappa_\sigma, \kappa_\rho)$ *has a prior. Then, the Bayes estimates are consistent almost surely. Namely,* $E[f(\theta)|\mathcal{F}_t^Z] \to f(\theta)$ *as* $t \to \infty$ *for bounded continuous function* f *when* $\mu - \frac{1}{2}(\sigma + \kappa_\sigma)^2 > 0$.

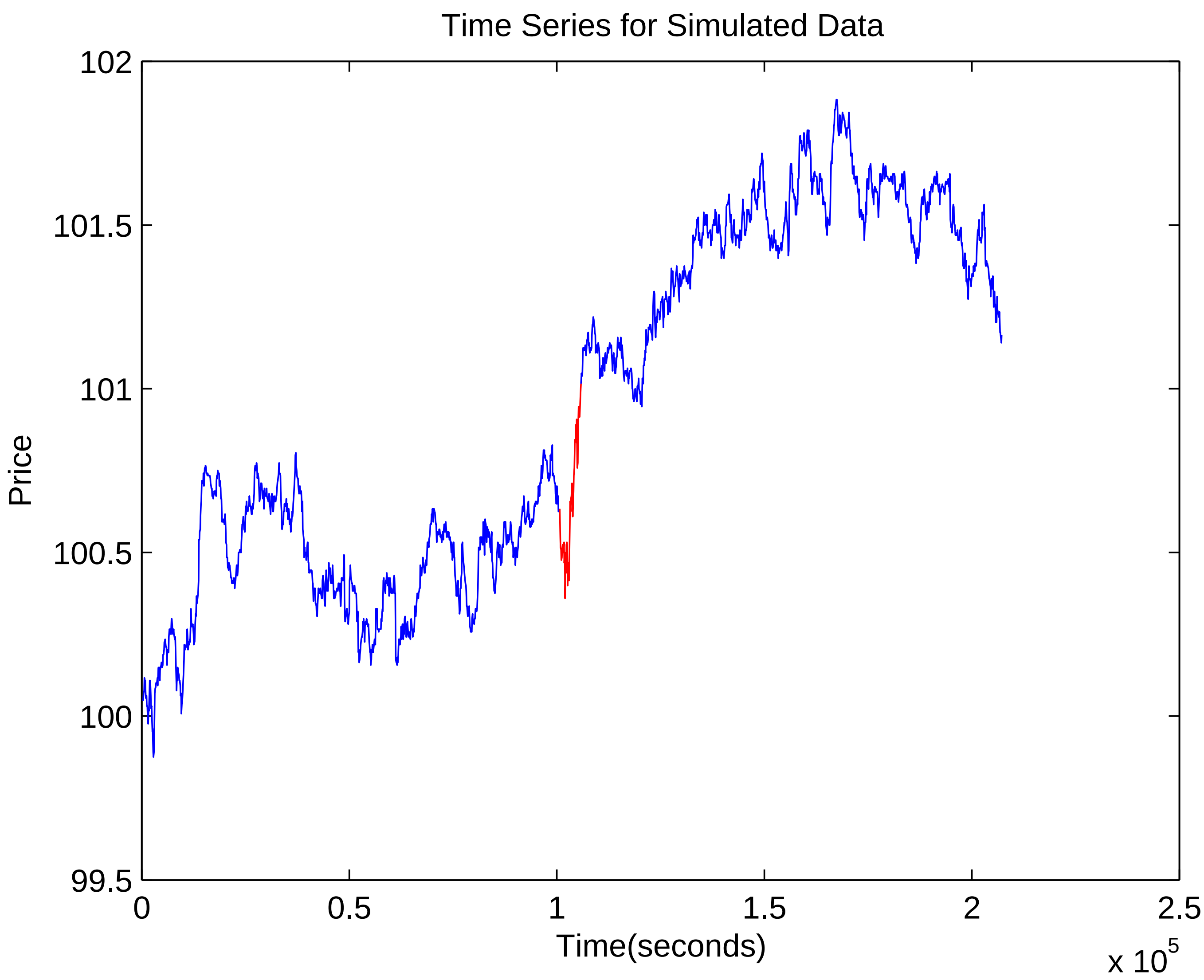
Remark: The standard condition "$\mu - \frac{1}{2}(\sigma + \kappa_\sigma)^2 > 0$" rules out bankruptcy.

Estimation Results: Cases (1) and (2)

Table 1: Annualized Bayes estimates for 10yr. Treasury UHF Data, 8/15/-12/31, 2000

Models	μ	σ	ρ	κ_σ	κ_ρ
Case I:	34.48%	4.92%	0.0503	-0.29%	-0.0109
$V_t = \pm 1$	(4.27%)	(0.05%)	(0.0045)	(0.04%)	(.0045)
Case II:	28.64%	4.94%	0.0486	-0.10%	n.a.
$V_t = \pm\sqrt{Vol}$	(8.17%)	(0.05%)	(0.0044)	(0.03%)	n.a.

Simulated Data: Case (3)

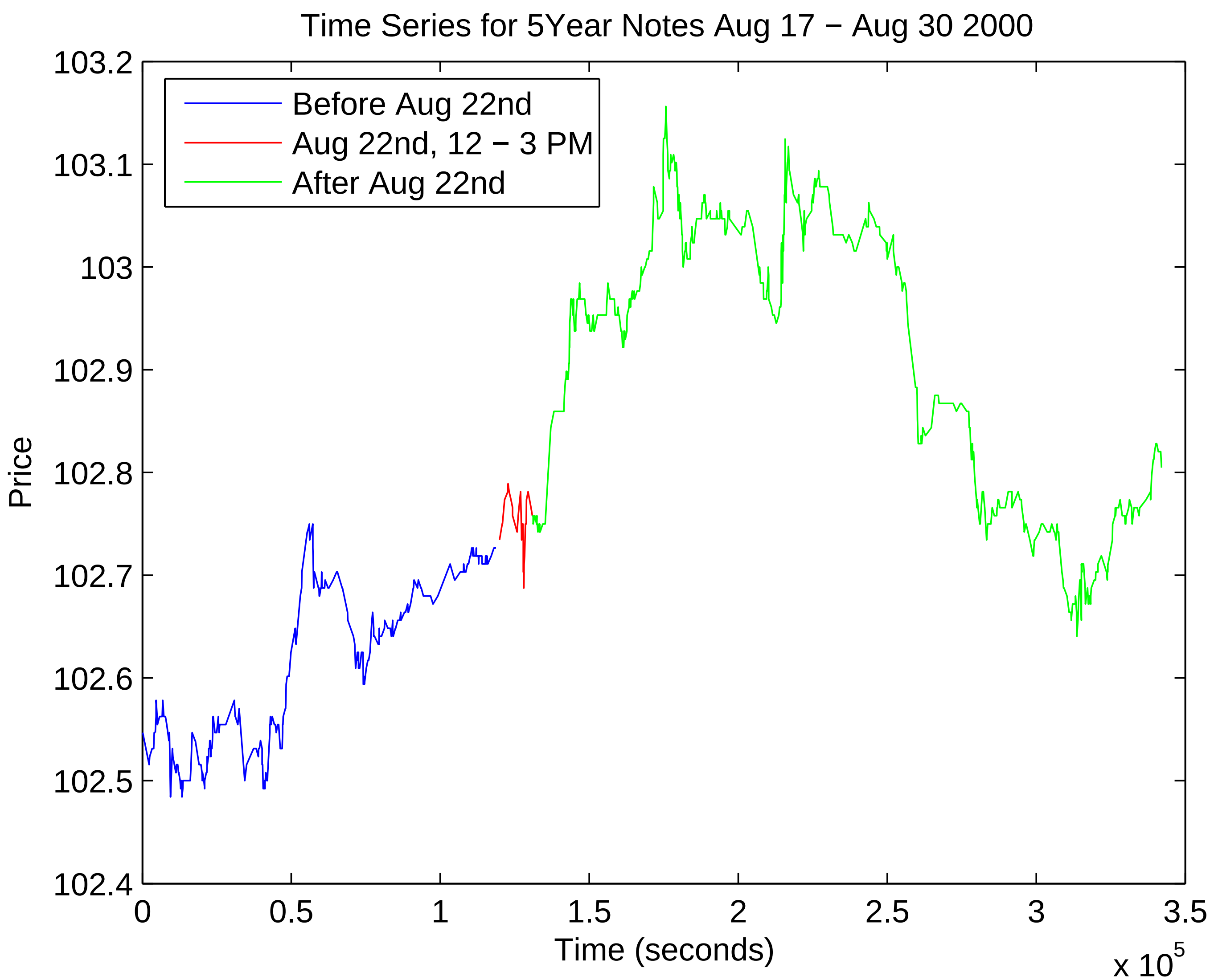


Bayes Est. for simulated data: Case (3)

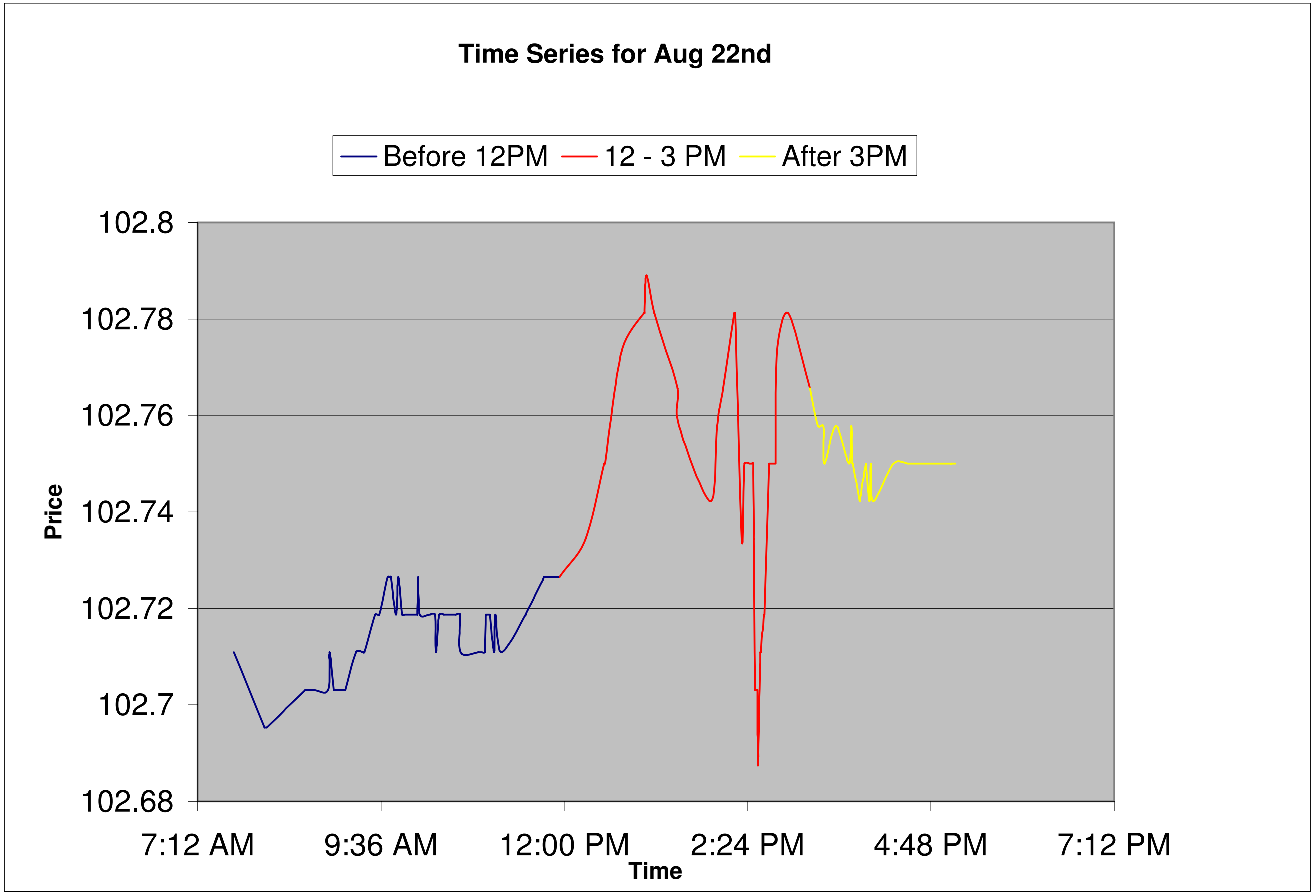
Table 2: Bayes estimates for 2050 simulated data when 50 data have extra volatility,

Models	μ	σ	ρ	κ_σ
True	5.00E-8	2.00E-5	0.05	3.00E-5
Bayes Est.	4.27E-8	2.00E-5	0.0477	2.86E-5
	(1.08E-8)	(2.89E-14)	(0.0056)	(6.58E-6)

Real Data: FOMC Period I



Real Data: FOMC Period II



Bayes Est. for Treasury data: Case (3)

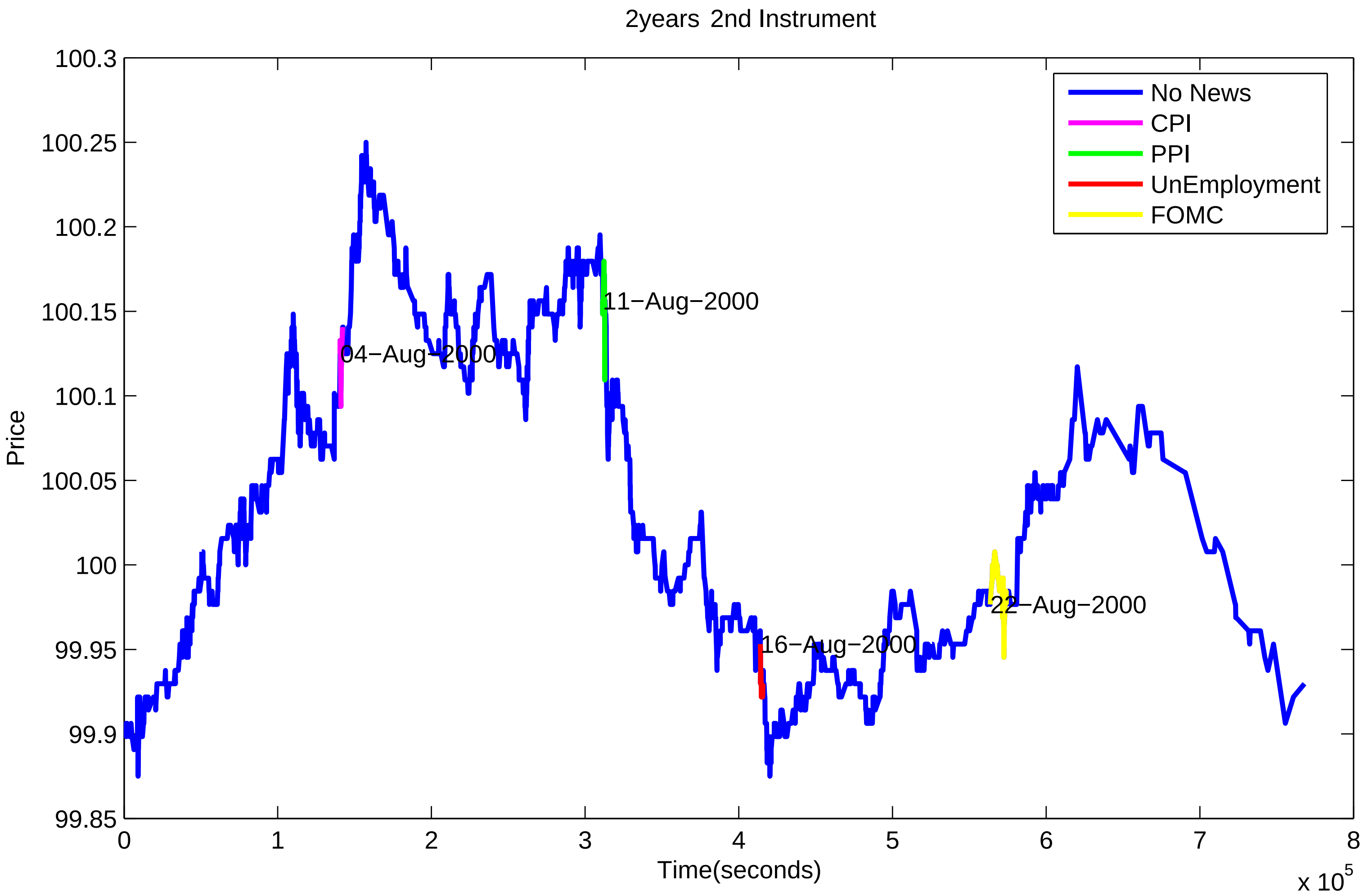
Table 3: Bayes estimates for August 17 - 30, 2000, 5yr notes

Models	μ	σ	ρ	κ_σ
Bayes Est.	8.51%	1.68%	0.022	1.93
	(7.21%)	(0.08%)	(0.0054)	(0.58%)

(1247 data with 40 marking V=1, for 12-3pm of August 22 when FOMC announcement time is around 2pm)

- These results are consistent with Fleming and Remolona (1999), Green (2004) and Brandt and Kavacejz (2004): Primary impact: Economic News; Secondary impact: order flow and liquidity.

Price and Economic Announcements



Bayes Est. (I) for Treasury data: Case (3)

Data	μ	σ	ρ	κ_σ
2yr: 7/2000	0.26%	0.93%	0.0054	3.32%
	(0.10%)	(0.03%)	(0.0073)	(0.16%)
2yr: 8/2000	0.06%	0.74%	0.0008	1.41%
	(0.05%)	(0.03%)	(0.0003)	(0.15%)
2yr: 9/2000	0.06%	1.04%	0.0014	2.16%
	(0.01%)	(0.02%)	(0.0002)	(0.29%)
2yr: 10/2000	0.09%	1.49%	0.0011	1.53%
	(0.02%)	(0.03%)	(0.0003)	(0.16%)
2yr: 11/2000	0.09%	0.99%	0.0009	1.95%
	(0.02%)	(0.03%)	(0.0001)	(0.30%)

Bayes Est. (II) for Treasury data: Case (3)

Data	μ	σ	ρ	κ_σ
2yr: 12/2000	0.12%	1.50%	0.0001	1.31%
	(0.04%)	(0.03%)	(0.0001)	(0.27%)
5yr: 7/1 - 11/16/2000	2.78%	2.71%	0.0204	4.19%
	(1.86%)	(0.03%)	(0.0029)	(0.03%)
5yr: 11/17 - 12/31/2000	27.66%	3.37%	0.0008	4.96%
	(4.62%)	(0.03%)	(0.0006)	(0.69%)
10yr: 7/1 - 8/12/2000	11.82%	4.81%	0.0276	8.60%
	(9.01%)	(0.14%)	(0.0051)	(0.16%)
10yr: 8/15 - 12/31/2000	13.77%	4.86%	0.0479	4.03%
	(7.33%)	(0.05%)	(0.0044)	(0.30%)

Conclusions and Future Works

- Financial applications on market microstructure theory
- Statistics and information theory:
  - Consistency, CLT for the estimators of parameters.
  - Mutual information and its rate and its applications
- Mathematical finance:
  - Option pricing and hedging, portfolio optimization, and utility maximization.
- Particle filtering or sequential Monte Carlo
  - Convergence, convergence rate and large deviation
- To allow long-range dependence by using fractional signal.

Related papers, real data examples, Fortran codes are available at
`http://mendota.umkc.edu/paper-tick.html`

Particle Filtering (*or SMC*)

Suppose parameters are known. To estimate the value process, X,

- **Simulate** 100 independent sample paths of X following GBM: $V_j(t)$, $j = 1, 2, \cdots, 100$, at each trading times.
- At some time, t_i, calculate the **importance weight** for each V_j:

$$w_i^j(V_j(t_i)) = L^j(t_i) = \prod_{k=1}^{n} \exp\left\{\int_{t_{i-1}}^{t_i} \log \lambda_k(V_j(s-), s-) dY_k(s)\right\}$$

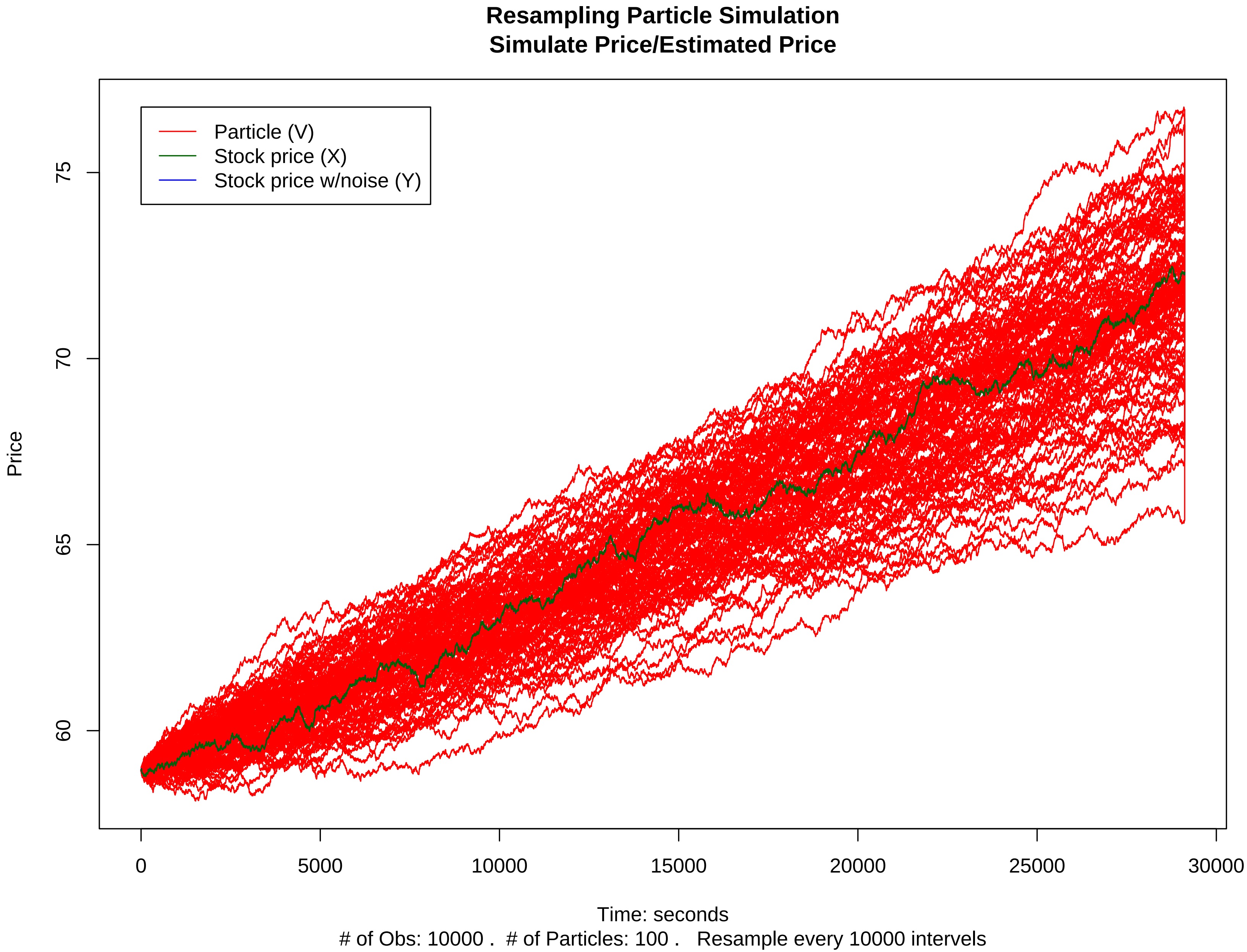
$$\times \prod_{k=1}^{n} \exp\left\{-\int_{t_{i-1}}^{t_i} \left[\lambda_k - 1\right] ds\right\}.$$

- **Resample** the sample paths according to the distribution proportional to the importance weights at times.

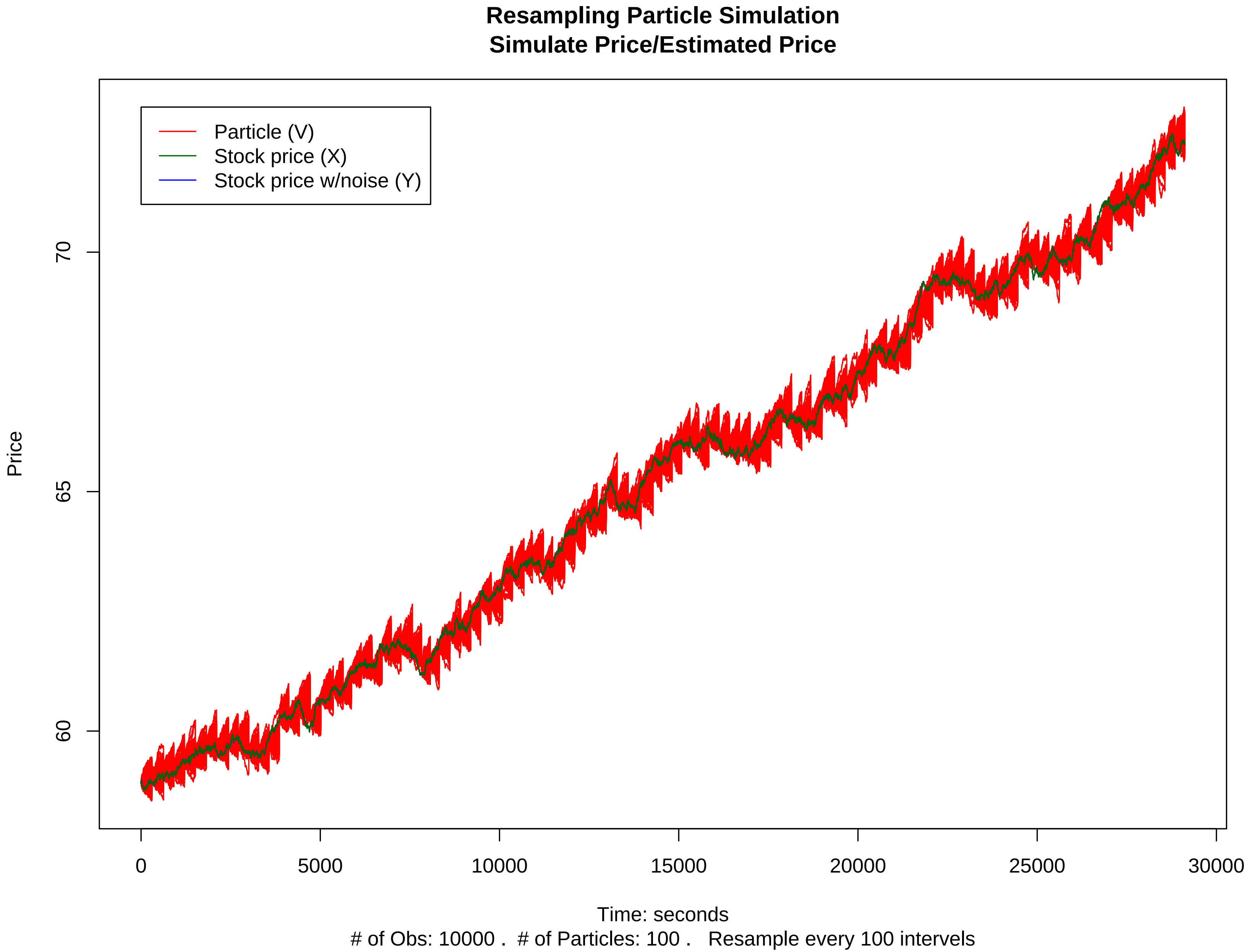
Or

- **Branch** each particle to a random number of particles proportional to the importance weights at times. (Xiong and Zeng 2006)

Simulation: No Resampling



Simulation: Resampling Every 100 Trades



Simulation: Resampling Every 10 Trades