# AN EXACT SOLUTION OF THE TERM STRUCTURE OF INTEREST RATE UNDER REGIME-SWITCHING RISK

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**Abstract.** Regime-switching risk has been recently studied in an general equilibrium setting and empirically documented as an significant factor in bond premium. In this paper, applying no arbitrage approach, we derive an exact solution of the term structure of interest rates in an essentially-affine-type model under regime switching risk.

**Keywords:** Term Structure Model, Regime Switching Risk, Marked Point Process, Affine Diffusion

## 1 Introduction

Much documented empirical evidence implies that the aggregate economy has recurrent shifts between distinct regimes of the business cycle (e.g Hamilton [17], and Diebold and Rudebusch [10]). These results have motivated the recent studies of the impact of regime shifts on the entire yield curve using dynamic term structure models. A common approach is to incorporate Markov-switching (or hidden Markov chains) into the stochastic processes of the pricing kernel and/or state variables. The regime-dependence indeed offers greater econometric flexibilities in empirical models of the term structure such as Bansal and Zhou [1]. However, as pointed out by Dai and Singleton [8], the risk of regime shifts is not priced in many of these models, and hence it does not contribute independently to bond risk premiums.

Without pricing the risk of regime shifts, the previous studies have essentially treated the regime shifts as an idiosyncratic risk that can be diversified away by bond investors. However, Bansal and Zhou [1] and Wu and Zeng

[22] have empirically shown that regimes are intimately related to the business cycle, suggesting a close link between the regime shift and aggregate uncertainties.

Extending the aforementioned strand of literature, Wu and Zeng [22] develop a dynamic term structure model under the systematic risk of regime shifts in a general equilibrium setting similar to Cox, Ingersoll and Ross [5] [6] (henceforth CIR). The model implies that bond risk premiums include two components under regime shifts. One is a regime-dependent risk premium due to diffusion risk as in the previous studies. The other is a regime-switching risk premium that depends on the covariations between the discrete changes in marginal utility and bond prices across different regimes. This new component of the term premiums is associated with the systematic risk of recurrent shifts in bond prices (or interest rates) due to regime changes and is an important factor that affects bond returns. Furthermore, we also obtain a closed-form solution of the term structure of interest rates under an affine-type model using the log-linear approximation similar to that in Bansal and Zhou [1]. The model is estimated by Efficient Method of Moments using monthly data on 6-month treasury bills and 5-year treasury bonds from 1964 to 2000. We find that the market price of regime-switching risk is highly significant and affects mostly the long-end of the yield curve. The regime-switching risk, as expected, accounts for a significant portion of the term premiums for long-term bond.

A drawback in Wu and Zeng [22] is that in a affine-type model, the closed form solution of the yield curve is obtained under log-linear approximation [1]. In this paper, using no arbitrage approach, we derive an exact solution of the term structure of interest rates in a more general essentially-affine-type model under regime-switching risk.

In the standard affine models (such as Duffie and Kan [13] and Dai and Singleton [7]), the market price of diffusion risk is proportional to the volatility of the state variable. Such a structure guarantees that the models satisfy a requirement of no-arbitrage: Risk compensation goes to zero as risk goes to zero. However, as Duffee [12] points out, this structure limits the variation of the compensations that investors anticipate to obtain when encountering a risk. More precisely, since the compensation is bounded below by zero, it cannot change sign over time. Duffee [12] argues that this is the main reason why the completely affine models fails at forecasting. He suggests a boarder class of essentially affine models to break the tight link between risk compensation and interest rate volatility. These more generally models are shown to have better forecasting ability than the standard affine models. In this paper, we introduce regime shifts into the class of essentially affine models. Our model

[1] Due to the natural of log-linear approximation, we conjecture the error bound should be in the order of $r^2$. Wu and Zeng [23] derived the close form for the multi-factor affine models with both jump and regime-switching risks using log-linear approximation.

with exact solution presented here further equips with regime-switching and should be more useful in forecasting future yields.

To the best of our knowledge, three other papers also presented exact solutions for regime switching term structure models. Two of them are continuous-time models. One is Landen [19] and the other is Dai and Singleton [8]. Landen [19] focused on the case under risk-neutral probability measure, she did not mention anything about the market price of regime switching risk. Dai and Singleton [8] surveyed the theoretical specification of dynamic term structure models. Moreover, they proposed a Gaussian affine-type model with regime-switching risk and constant volatility within each regime. In our model, we allow for stochastic volatilities in each regime and the diffusion risk is in an essentially affine form. The third one is Dai, Singleton and Yang [9]. They develop and empirically implement a discrete-time Gaussian dynamic term structure model with priced factor and regime-shift risks.

The rest of the paper is organized as follows. Section 2 presents a simpler expressive form of regime shifting using marked point process (or random measure) approach. Section 3 first develops a framework for the term structure of interest rates with regime-switching risk using the no arbitrage approach. Section 4 specifies a essentially-affine-type model with regime switching risk and derives an exact solution. Section 5 concludes with some future research topics.

## 2 A New Representation for Modeling Regime Shift

In the literature of interest rate term structure, there are three approaches to model regime shifting process. The first approach is the *Hidden Markov Model*, summarized in the book of Elliott et. al. [15], and its application to the term structure can be found in Elliott and Mamon [16]. The second approach is the *Conditional Markov Chain*, discussed in Yin and Zhang [24], and its applications to the term structure are in Bielecki and Rutkowski ([2],[3]). The third approach is the *Marked Point Process* or the *Random Measure* approach as in Landen [19]. Due to its notational simplicity, here, we follow the third approach but propose a new and simpler representation. In Landen [19], the mark space is a product space of regime $E = \{(i,j) : i \in \{1,...,N\}, j \in \{1,2,...,N\}, i \neq j\}$, including all possible regime switchings. Below, we simplify the mark space to the space of regime only and consequently simplify the corresponding random measure as well as the equation for $s(t)$ or $s_t$, which is defined as the most recent regime.

There are two steps to obtain the simple expression of $s(t)$.

**Step 1:** we define a random counting measure. Let the mark space $U = \{1,2,...,N\}$ be all possible regimes with the power $\sigma$-algebra, and $u$ be a generic point in $U$. Let $A$ be a subset of $U$. Let $m(t,A)$ counts the cumulative number of entering a regime that belongs to $A$ during the time $(0,t]$. For example, $m(t, \{u\})$ counts the cumulative number of entering regime $u$ during $(0, t]$. Note that $m$ is a random counting measure. Let $\eta$ be the usual counting measure on $U$. Then, $\eta$ has the following two properties: For $A \in U$, $\eta(A) = \int I_A \eta(u)$ (i.e. $\eta(A)$ counts the number of elements in A) and $\int_A f(u)\eta(u) = \sum_{u \in A} f(u)$.

A marked point process or a random measure is uniquely characterized by its stochastic intensity kernel,[2]. Let $x(t)$ denote a state variable to be defined later. Then, the stochastic intensity kernel of $m(t, \cdot)$ can be defined as

$$\gamma_m(dt, du) = h(u; x(t-), s(t-))\eta(du)dt, \tag{1}$$

where $h(u; x(t-), s(t-))$ is the conditional regime-shift (from regime $s(t-)$ to $u$) intensity at time $t$ (we assume $h(u; x(t-), s(t-))$ is bounded). Heuristically, $\gamma_m(dt, du)dt$ can be thought of as the conditional probability of shifting from Regime $s(t-)$ to Regime $u$ during $[t, t+dt)$ given $x(t-)$ and $s(t-)$. Then, $\gamma_m(t, A)$, the compensator of $m(t, A)$, can be written as

$$\begin{aligned}\gamma_m(t, A) &= \int_0^t \int_A h(u; x(\tau-), s(\tau-))\eta(du)d\tau \\ &= \sum_{u \in A} \int_0^t h(u; x(\tau-), s(\tau-))d\tau.\end{aligned}$$

**Step 2:** We are in the position to present the integral and differential forms for the evolution of regime, $s(t)$, using the random measure defined above. First, the integral form is:

$$s(t) = s(0) + \int_{[0,t] \times U} (u - s(\tau-))m(d\tau, du). \tag{2}$$

Note that $m(d\tau, du)$ is zero most of time and only becomes one at regime-switching time $t_i$ with $u = s(t_i)$, the new regime at time $t_i$. Observe that the above expression is but a telescoping sum: $s(t) = s(0) + \sum_{t_i < t}(s(t_i) - s(t_{i-1})$. Second, the differential form is:

$$ds(t) = \int_U (u - s(t-))m(dt, du). \tag{3}$$

To see the above differential equation is valid, assuming there is a regime switching from $s(t-)$ to $u$ occurs at time $t$, then $s(t) - s(t-) = (u - s(t-))$ implying $s(t) = u$.

These two forms are crucial in the following two sections.

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[2] See Last and Brandt [20] for detailed discussion of marked point process, stochastic intensity kernel and related results.

# 3 The Model

## 3.1 Two State Variables

We assume there are two state variables. One describes the regime change, $s(t)$ or $s_t$, which stands the most recent regime. As described in Section 2, $s_t$ follows (2) or (3). The other state variables, $x_t$, is described by a diffusion

$$dx_t = \mu(x_t, s_t)dt + \sigma(x_t, s_t)dW_t \tag{4}$$

where the drift term and the diffusion term are in general time-varying and regime-dependent, and $W_t$ is a standard Brownian motion.

The instantaneous short-term interest $r_t$ is a linear function of $x_t$ given $s_t$

$$r_t = \psi_0(s_t) + \psi_1 x_t \tag{5}$$

where $\psi_0(s_t)$ is a constant depending on regime but $\psi_1$ is not. When $\psi_1$ is also regime-dependent, we can not obtain an exact solution.

## 3.2 Pricing Kernel

Under certain technical conditions, absence of arbitrage is sufficient for the existence of the pricing kernel (see Harrison and Kreps [18]). We further specify the pricing kernel $M_t$ as

$$\begin{aligned} \frac{dM_t}{M_{t-}} =& - r_{t-}dt - \lambda_D(x_t, s_t)dW_t \\ & - \int_U \lambda_S(u; x_t, s_{t-})[m(dt, du) - \gamma_m(dt, du)] \end{aligned} \tag{6}$$

where $\lambda_D(x_t, s_t)$ is the of market price of diffusion risk, which is also regime-dependent; and $\lambda_S(u; x_{t-}, s_{t-})$ is the market price of regime-switching (from regime $s_{t-}$ to regime $u$) risk given $x_t$ and $s_{t-}$.

Note that the explicit solution for $M_t$ can be obtained by Doleans-Dade exponential formula (Protter [21]) as the following:

$$\begin{aligned} M_t = & \left(e^{-\int_0^t r_\tau d\tau}\right) \left(e^{-\int_0^t \lambda_D(x_\tau, s_\tau)dW(\tau) - \frac{1}{2}\int_0^t \lambda_D^2(x_\tau, s_\tau)d\tau}\right) \times \\ & \left(e^{\int_0^t \int_U \lambda_S(u; s_{\tau-}, s_{\tau-})\gamma_m(d\tau, du) + \int_0^t \int_U \log(1-\lambda_S(u; x_{\tau-}, s_{\tau-}))m(d\tau, du)}\right) \end{aligned} \tag{7}$$

## 3.3 The Risk-Neutral Probability Measure

The specifications above complete the model for the term structure of interest rates, which can be solved by a change to the risk-neutral probability measure. We first obtain the following two lemmas. The first lemma characterizes the equivalent martingale measure under which the interest rate term structure is determined. The second lemma specifies the dynamic of the short rate and the regime under the equivalent martingale measure.

**Lemma 1.** *For fixed $T > 0$, the equivalent martingale measure* $\mathbf{Q}$ *can be defined by the Radon-Nikodym derivative below*

$$\frac{dQ}{dP} = \xi_T/\xi_0$$

*where for* $t \in [0, T]$

$$\xi_t = \left(e^{-\int_0^t \lambda_D(x_\tau, s_\tau)dW(\tau) - \frac{1}{2}\int_0^t \lambda_D^2(x_\tau, s_\tau)d\tau}\right) \times \left(e^{\int_0^t \int_U \lambda_S(u; x_\tau, s_{\tau-})\gamma_m(d\tau, du) + \int_0^t \int_U \log(1-\lambda_S(u; x_\tau, s_{\tau-}))m(d\tau, du)}\right) \tag{8}$$

*provided* $\lambda_D$, $\lambda_S$ *and* $h$ *in* $m(t, A)$ *are all bounded in* $[0, T]$.

*Proof.* Obviously, $\xi_t > 0$ for all $0 \le t \le T$. By Doleans-Dade exponential formula, $\xi_t$ can be written in stochastic differential equation form as

$$\frac{d\xi_t}{\xi_t} = -\lambda_D(x_t, s_t)dW_t - \int_U \lambda_S(u; x_t, s_{t-})[m(dt, du) - \gamma_m(dt, du)]. \tag{9}$$

Since $W_t$, and $m(t, A) - \gamma_m(t, A)$ are martingales under $P$, $\xi_t$ is a local martingale.

Since $\xi_t$ is a $\mathbf{P}$-local martingale and $\xi_0 = 1$, it suffices to show that $E([\xi]_t) < \infty$ to obtain $E(\xi_t) = 1$ for all $t$, because $\xi$ becomes a martingale if $E([\xi]_t) < \infty$ for all $t$, where $[\xi]_t$ is the quadratic variation process of $\xi$. Let

$$K_t = -\int_0^t \lambda_D(x_\tau, s_\tau)dW_\tau - \int_0^t \int_U \lambda_S(u; x_\tau, s_{\tau-})[m(dt, du) - \gamma_m(d\tau, du)]$$

By assumption, we suppose that $|\lambda_D(x_t, s_t)| \le C_D$ for all $x_t$ and $s_t$, and $|\lambda_S(u; x_t, s_{t-})| \le C_S$ and $h(u; x_t, s_{t-}) \le C_h$ for all $u, x_t$ and $s_{t-}$. Using the properties of quadratic variation for semimartingale (see Section 2.6 of Protter [21]), we have

$$[K]_t = \int_0^t \lambda_D^2(x_\tau, s_\tau)d\tau + \int_0^t \int_U \lambda_S^2(u; x_\tau, s_{\tau-})m(d\tau, du).$$

Observe that for $t \le T$,

$$\int_0^t \lambda_D^2(x_\tau, s_\tau)d\tau \le C_D^2\ T.$$

and

$$0 < \sum_u \lambda_S^2(u; x_t, s_{t-})h(u; x_t, s_{t-}) \le N\ C_S^2\ C_h.$$

Then,

$$
\begin{aligned}
E([\xi]_t) &= E\int_0^t \xi_{\tau-}^2 d[K]_\tau \\
&= E\left\{\int_0^t \xi_{\tau-}^2 \lambda_D^2(x_\tau, s_\tau)d\tau + \int_0^t \xi_{\tau-}^2 \int_U \lambda_S^2(u;x_\tau,s_{\tau-})m(d\tau,du)\right\} \\
&\le E\left\{C_D^2\int_0^t \xi_\tau^2 d\tau + \int_0^t \xi_{\tau-}^2 \int_U \lambda_S^2(u;x_\tau,s_{\tau-})m(d\tau,du)\right\} \\
&\le E\left\{C_D^2\int_0^t \xi_\tau^2 d\tau + \int_0^t \xi_{\tau-}^2 \int_U \lambda_S^2(u;x_\tau,s_{\tau-})\gamma_m(d\tau,du)\right\} \\
&\le E\left\{C_D^2\int_0^t \xi_{\tau-} d\tau + \int_0^t \xi_{\tau-}^2 \int_U \lambda_S^2(u;x_\tau,s_{\tau-})h(u;x_\tau,s_{\tau-})\eta(du)d\tau\right\} \\
&\le E\left\{C_D^2\int_0^t \xi_{\tau-} d\tau + \int_0^t \xi_{\tau-}^2 \sum_u \lambda_S^2(u;x_\tau,s_{\tau-})h(u;x_\tau,s_{\tau-})d\tau\right\} \\
&\le E\left\{C_D^2\int_0^t \xi_\tau^2 d\tau + N\; C_S^2\; C_h \int_0^t \xi_{\tau-}^2 d\tau\right\} \\
&\le C^* \int_0^t E(\xi_\tau^2)d\tau
\end{aligned}
\tag{10}
$$

for $C^* = \max(C_D^2, N\; C_S^2\; C_h)$. By the same boundedness and from the direct calculation of expected values under normal and Poissons, we obtain

$$E(\xi_t^2) < C^{**}$$

for some constant $C^{**}$. This implies $E([\xi_t]) < \infty$ for all $t$ and hence, $\xi_t$ is a martingale. □

**Lemma 2.** *Under the risk-neutral probability measure* $\mathbf{Q}$*, the dynamics of state variables,* $x_t$ *and* $s_t$*, are given by the following stochastic differential equations respectively*

$$dx_t = \tilde{\mu}(x_t, s_t)dt + \sigma(x_t, s_t)d\tilde{W}_t \tag{11}$$

$$dx_t = \int_U (u - s_{t-})\tilde{m}(dt, du) \tag{12}$$

*where* $\tilde{\mu}(x_t, s_t) = \mu(x_t, s_t) - \sigma(x_t, s_{t-})\lambda_D(x_t, s_t)$*;* $\tilde{W}_t$ *is a standard Brownian motion under* $\mathbf{Q}$*;* $\tilde{m}(t, A)$*, the corresponding marked point process of* $m(t, A)$ *under* $\mathbf{Q}$*, has the intensity matrix* $\tilde{H}(u; x_{t-}, s_{t-}) = \{\tilde{h}(u; x_{t-}, s_{t-})\} = \{h(u; x_{t-}, s_{t-})\ (1 - \lambda_S(u; x_{t-}, s_{t-}))\}$*. The compensator of* $\tilde{m}(t, A)$ *under* $\mathbf{Q}$ *becomes*

$$\gamma_{\tilde{m}}(dt, du) = (1 - \lambda_S(u; x_{t-}, s_{t-}))\gamma_m(dt, du) = \tilde{h}(u; x_{t-}, s_{t-})\eta(du)dt,$$

*Proof.* Applying Girsanov's Theorem on the change of measure for Brownian motion, we have $\tilde{W}_t = W_t - \int_0^t \lambda_D(x_\tau, s_\tau)d\tau$ is a standard Brownian motion under $\mathbf{Q}$. This allows us to obtain $\tilde{\mu}(x_t, s_t) = \mu(x_t, s_t) - \sigma(x_t, s_t)\lambda_D(x_t, s_t)$.

Since the marked point process, $\mu(t, A)$, is actually a collection of $N(N-1)$ conditional Poisson processes, by applying Girsanov's Theorem on conditional Poisson process (for example, see Theorem T2 and T3 in Chapter 6 of Bremaud [4]), the conditional Poisson process with intensity, $h(u; x_t, s_{t-})$, under $\mathbf{P}$, becomes the one with intensity, $h(u; x_t, s_{t-})(1 - \lambda_S(u; x_t, s_{t-}))$ under $\mathbf{Q}$. Then, the result follows. □

### 3.4 The Term Structure of Interest Rates

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration generated by $W$ and $m(t, \cdot)$. In the absence of arbitrage, the price at time $t-$ of a default-free pure discount bond that matures at $T$, $P(t-, T)$, can be obtained as,

$$P(t-, T) = E^{\mathbf{Q}}_{t-}(e^{-\int_t^T r_\tau d\tau}) = E^{\mathbf{Q}}\{e^{-\int_t^T r_\tau d\tau}|\mathcal{F}_t\} = E^{\mathbf{Q}}\{e^{-\int_t^T r_\tau d\tau}|x_t, s_{t-}\} \tag{13}$$

with the boundary condition $P(T-, T) = P(T, T) = 1$ and the last equality comes from the Markov property of $(x_t, s_t)$.

Therefore, we can let $P(t-, T) = F(t-, x_t, s_{t-}, T) = F(t-, x, s, T)$ where $x = x_{t-}$ and $s = s_{t-}$. The following proposition gives the partial differential equation characterizing the bond price.

**Proposition 1.** *The price of the default-free pure discount bond* $F(t-, x, s, T)$ *defined in* (13) *satisfies the following partial differential equation*

$$\frac{\partial F}{\partial t} + \tilde{\mu}(x, s)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(x, s)\frac{\partial^2 F}{\partial x^2} + \int_U \Delta_S F\ \tilde{h}(u; x, s)\eta(du) = rF \tag{14}$$

*with the boundary condition* $F(T-, x, s, T) = F(T, x, s, T) = 1$ *and* $\Delta_S F = F(t, x, u, T) - F(t-, x, s, T)$.

*Proof.* It basically comes from Feynman-Kac formula. Or, intuitively, the above result is obtained by applying Ito's formula for semimartingale (Protter [21]) under measure $\mathbf{Q}$ to $F(t, x, s, T)$

$$\begin{aligned} dF =& (\frac{\partial F}{\partial t} + \tilde{u}\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2})dt + \sigma\frac{\partial F}{\partial x}d\tilde{W}_t \\ &+ [F(t, x_t, s_t, T) - F(t-, x_{t-}, s_{t-}, T)] \end{aligned} \tag{15}$$

Since $x(t)$ is continuous, the last term can be expressed by

$$\int_U \Delta_S F\ \tilde{m}(dt, du).$$

Note that the above term can be made as a martingale by subtracting its own compensator, which is added back in $dt$ term. Note that $\gamma_{\tilde{m}}(dt, du) = \tilde{h}(u; x(t-), s(t-))\eta(du)dt$. Therefore we can have the following equation for $F(t-, x, s, T)$

$$\begin{aligned} dF = \{&\frac{\partial F}{\partial t} + \tilde{u}\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} + \int_U \Delta_S F \; \tilde{h}(u; x, s)\eta(du)\}dt \\ &+ \sigma\frac{\partial F}{\partial x}d\tilde{W}_t + \int_U \Delta_s F \; [\tilde{m}(dt, du) - \gamma_{\tilde{m}}(dt, du)] \end{aligned} \tag{16}$$

Since no arbitrage implies that the instantaneous expected returns of all assets should be equal to the short-term interest rate under the risk-neutral measure, equation (14) then follows by matching the coefficient of the $dt$ term of (16) with $rF$. □

## 4 A Tractable Specification with Exact Solution

In general equation (14) doesn't admit a closed form solution for the bond price. In this section, We consider a tractable specification: affine term structure of interest rates with regime switching and regime-switching risk.

### 4.1 Affine Regime-Switching Models

Duffie and Kan [13] and Dai and Singleton [7], among other, have detailed discussions of completely affine term structure models under diffusions. Duffie, Pan and Singleton [14] deals with general asset pricing under affine jump-diffusions. Duffee [12] presented a class of *essentially affine models* and Duarte [11] introduced *semi-affine models*. Bansal and Zhou [1] and Landen [19] all use affine structure for their regime switching models. Following this literature, we make the following parametric assumptions

**Assumption 1** *The diffusion components of $x_t$, as well as those in the Markov switching process $s_t$ all have an affine structure. In particular,*

(1) $\mu(x_t, s_t) = a_0(s_t) + a_1(s_t)x_t$,

(2) $\sigma(x_t, s_t) = \sqrt{\sigma_0(s_t) + \sigma_1 x_t}$,

(3) $h(u; x_t, s_{t-}) = \exp\{h_0(u; s_{t-}) + h_1(u; s_{t-})x_t\}$;

(4) $\lambda_D(x_t, s_t) = \dfrac{\lambda_0(s_t) + \lambda_1 x_t + a_1(s_t)x_t}{\sqrt{\sigma_0(s_t) + \sigma_1 x_t}}$,

(5) $1 - \lambda_S(u; x_t, s_{t-}) = e^{\theta(u; s(t-))}/h(u; x_t, s_{t-})$.

The first three assumptions are related to the two state processes. For the diffusion state process, we assume that the drift term and the volatility term are all affine functions of $x_t$ with regime-dependent coefficients. Then, $x(t)$ becomes

$$dx = (a_0(s) + a_1(s)\,x)\,dt + \sqrt{\sigma_0(s) + \sigma_1\,x}\,dW_t \tag{17}$$

We further assume that the log intensity of regime shifts is an affine function of the short term rate $x_t$. This assumption ensures the positivity of the intensity function and also allows the transition probability to be time-varying.[3]

The last two assumptions deal with the market prices of risk. In the completely affine models, the market price of diffusion risk is proportional to the volatility of the state variable $x_t$. Such a structure guarantees that the models satisfy a requirement of no-arbitrage: Risk compensation goes to zero as risk goes to zero. However, since variances are nonnegative, this structure limits the variation of the compensations that investors anticipate to obtain when encountering a risk. More precisely, since the compensation is bounded below by zero, it cannot change sign over time. This restriction, however, is relaxed in the essentially affine models of Duffee [12].

Following this literature, we also assume that the market price of the diffusion risk is in the form of essentially affine, but we extend to the case with regime switching with small twists. Namely, we assume the diffusion risk is a sum of regime-dependent linear combination of $x_t$ and non-regime-dependent scaler of $x_t$ divided by the diffusion coefficient. For the market price of regime switching risk, we assume that a regime-switching dependent constant divided by the intensity of regime switching. We choose these forms of market prices because we may obtain a closed-form solution to the bond prices.

Under these parameterizations of the market prices of risk, the state process $x_t$ and the Markov chain $s_t$ preserve the affine structure. In particular, under the risk-neutral measure $\mathbf{Q}$ the drift term $\tilde{\mu}(s,r)$, and the *log* of regime switching intensity $\tilde{h}(u;x,s)$ are affine functions of the state $x$ with regime dependent coefficients. Precisely, under the risk-neutral measure $\mathbf{Q}$,

$$\begin{aligned} dx &= (a_0(s) + a_1(s)\,x)\,dt + \sqrt{\sigma_0(s) + \sigma_1\,x}\,d\tilde{W}_t - [\lambda_0(s) + \lambda_1 x + a_1(s)x]dt \\ &= [a_0(s) - \lambda_0(s) - \lambda_1\,x]\,dt + \sqrt{\sigma_0(s) + \sigma_1\,x}\,d\tilde{W}_t. \end{aligned}$$

So, the coefficient

$$\tilde{\mu}(x,s) = a_0(s) - \lambda_0(s) - \lambda_1\,x$$

and $\sigma(x,s)$ remains the same. And

$$\tilde{h}(u;x,s(t-)) = e^{\theta(u;s(t-))}.$$

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[3] A more general specification is to allow duration-dependence as well. However a closed-form solution for the yield curve may not be attainable.

Then, we can solve for the term structure of interest rates and obtain the closed form solution as follows:

**Theorem 2.** *Under the Assumption 1, the price at time $t$ of a risk-free pure discount bond with maturity $\tau$ is given by $f(s(t-), x(t), \tau) = e^{A(\tau, s_t) + B(\tau) x_t}$ and the $\tau$-period interest rate is given by $R(t-, \tau) = -A(\tau, s_{t-})/\tau - B(\tau) x_t/\tau$. With $s = s(t-)$, $A(\tau, s)$ and $B(\tau)$ are determined by the following ordinary differential equations*

$$-\frac{\partial B(\tau)}{\partial \tau} - \lambda_1 B(\tau) + \frac{1}{2}\sigma_1 B^2(\tau) = \psi_1 \tag{18}$$

*and*

$$\begin{aligned} &-\frac{\partial A(\tau, s)}{\partial \tau} + [a_0(s) - \lambda_0(s)]B(\tau) + \frac{1}{2}\sigma_0(s) B^2(\tau) \\ &+ \int_U \left[e^{\Delta_S A(\tau, s)} - 1\right] e^{\theta(u;s)} \eta(du) = \psi_0(s) \end{aligned} \tag{19}$$

*with boundary conditions $A(0, s) = 0$ and $B(0) = 0$, where $\Delta_S A = A(\tau, u) - A(\tau, s)$.*

*Proof.* Without loss of generality, let the price at time $t-$ of a pure-discount bond that will mature at $T$ be given as

$$F(t-, s(t-), x(t), T) = f(s(t-), x(t), \tau) = e^{A(\tau, s(t-)) + B(\tau) x(t)}$$

where $\tau = T - t$ and $A(0, s) = 0$, $B(0, s) = 0$.

Then, the basic idea is to calculate the derivatives of the above $F$, substitute them in Eq.(14), and match the coefficients of $x$.

Observe that

$$\frac{\partial F}{\partial \tau} = F\left(-\frac{\partial A(\tau, s)}{\partial \tau} - \frac{\partial B(\tau)}{\partial \tau} x\right), \quad \frac{\partial F}{\partial x} = F B(\tau), \quad \frac{\partial^2 F}{\partial x^2} = F B^2(\tau),$$

$$\tilde{h}(u; x(t), s(t-)) = h(u; x(t), s(t-))(1 - \lambda_S(u; x(t), s(t-)) = e^{\theta(u; s(t-))},$$

$$F_S = F(e^{\Delta_S A} - 1)$$

where $\Delta_S A = A(\tau, u) - A(\tau, s)$, and recall that

$$r = \psi_0(s) + \psi_1 x.$$

With some simplifications and letting $s = s(t-)$, Proposition 1 then implies

$$\begin{aligned} \psi_0(s) + \psi_1 x = &-\frac{\partial A(\tau, s)}{\partial \tau} - \frac{\partial B(\tau)}{\partial \tau} x + [a_0(s) - \lambda_0(s) - \lambda_1 x] B(\tau) \\ &+ \frac{1}{2}[\sigma_0(s) + \sigma_1 x] B^2(\tau) + \int_U \left(e^{\Delta_S A} - 1\right) e^{\theta(u;s)} \eta(du) \end{aligned} \tag{20}$$

Then, Theorem 2 follows by matching the coefficients on $x$ on both side of the above equation. □

The above model extends the existing literature on the term structure of interest rates under regime shifts in several ways. While Landen [19] provided an exact solution to the yield curve only under risk-neutral probability measure, she was silent on the market price of regime switching risk. In the survey paper of Dai and Singleton [8] proposed a Gaussian affine-type model with regime-switching risk and constant volatility within each regime. In our model, we allow for stochastic volatilities in each regime and our diffusion risk is in a essentially affine form. In the case of Bansal and Zhou [1], the risk of regime shifts is not priced neither, and they had to rely on log-linear approximation to obtain closed form solution for bond pricing.

Finally, we examine the expected excess return on a long term bond over the short rate implied by our model.

**Corollary 1.** *Under the assumptions of Theorem 2, the expected excess return on a long term bond over the short rate is given by*

$$E_t\left(\frac{df_t}{f_{t-}}\right) - r_t dt = [\lambda_0(s) + \lambda_1 x + a_1(s)x]B(\tau)dt + \int_U \left(e^{\Delta_S A} - 1\right)\left(e^{h_0(u;s)+h_1(u;s)x} - e^{\theta(u;s)}\right)\eta(du)dt. \tag{21}$$

*Proof.* Similary, let the price at time $t-$ of a pure-discount bond that will mature at $T$ be given as

$$F(t-, s(t-), x(t), T) = f(s(t-), x(t), \tau) = e^{A(\tau, s(t-)) + B(\tau)x(t)}$$

where $\tau = T - t$ and $A(0, s) = 0$, $B(0, s) = 0$.

Applying It's formula to $F(t-, s(t-), x(t), T)$ under the physical measure $\mathbf{P}$, we obtain the following similar to Eq.(16)

$$\begin{aligned} dF = \{&\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} + + \int_U \Delta_S F\ h(u; x, s)\eta(du)\}dt \\ &+ \sigma\frac{\partial F}{\partial x}dW_t + \int_U \Delta_s F\ [m(dt, du) - \gamma_m(dt, du)] \end{aligned} \tag{22}$$

Applying Proposition 1, we want to make the coefficient of $dt$ term equal to $rF$ by subtracting and adding. Using Lemma 2, noting the last two terms are martingales, and taking conditional expectation with some simplifications, we obtain:

$$\begin{aligned} E_t\left(\frac{dF}{F}\right) - r_t dt = &\ \sigma(x, s)\lambda_D(x, s)\frac{\partial F}{\partial x}/F dt \\ &+ \int_U \frac{\Delta_S F}{F} h(u; x, s)\ \lambda_S(u; x, s)\eta(du)dt \end{aligned} \tag{23}$$

With simplifications, Assumption 1 implies that the above equation becomes Eq.(21). □

The first term on the right hand side of equation (21) is interpreted as the diffusion risk premium in the literature, and the second can be analogously defined as the regime-switching risk premium. The equation shows that introducing the dependence of the market prices of diffusion on $s_t$ adds more flexibility to the specification of the risk premium. Bansal and Zhou [1] points out that it is mainly this feature of the regime switching model that provides improved goodness-of-fit over the existing term structure models. On the other hand, (21) also shows that if the term structure exhibits significant difference across regimes ($\Delta_s A \neq 0$), there is an additional source of risk due to regime shifts and it should also be priced ($e^{h_0(u;s)+h_1(u;s)x} - e^{\theta(u;s)}$) in the term structure model. Introducing the regime switching risk not only can add more flexibilities to the specification of time-varying bond risk premiums, but also can be potentially important in understanding the bond risk premiums over different holding periods.

## 5 Conclusions

This paper first presents a new marked point process representation of regime change using a random measure. We apply this new representation to specify a term structure model of interest rate with regime-switching risk. We derive an exact solution for the yield curve in an essentially-affine specification.

With this exact solution, we can further estimate the model by Efficient Method of Moments as in Wu and Zeng [22] and quantify the regime-switching risk and its impact on yield curves. Other important topics include the implications and impacts of regime-switching risk on bond derivatives, and on investors' optimal portfolio choice problem. Also, more studies are needed on the empirical evidence of regime-switching risk in interest rates. These topics are left for future research.

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