

1 International Journal of Theoretical and Applied Finance
 Vol. 8, No. 7 (2005) 1–31
 3 © World Scientific Publishing Company



5 **A GENERAL EQUILIBRIUM MODEL OF THE TERM
 STRUCTURE OF INTEREST RATES UNDER
 REGIME-SWITCHING RISK**

7 SHU WU

9 *Department of Economics at The University of Kansas
 Summerfield Hall 213, Lawrence, KS 66045, USA
 shuwu@ku.edu*

11 YONG ZENG

13 *Department of Mathematics and Statistics at
 University of Missouri at Kansas City
 Kansas City, MO 64110, USA
 zeng@mendota.umkc.edu*

15 Received 10 June 2004
 17 Accepted 28 February 2005

19 This paper develops a general equilibrium model of the term structure of interest rates in
 the presence of the systematic risk of regime shifts. The model elucidates the economic
 21 nature of the regime-shift risk premium and introduces a new source of time-variation in
 bond returns. A closed-form solution for the term structure of interest rates is obtained
 23 under an affine model using log-linear approximation. The model is estimated by Efficient
 Method of Moments. The regime-switching risk is found to be statistically significant and
 mostly affect the long-end of the yield curve.

25 *Keywords:* Term structure; general equilibrium; Markov regime shifts.

1. Introduction

27 There exists a large body of empirical evidence suggesting that the aggregate econ-
 29 omy is characterized by periodic shifts between distinct regimes of the business
 cycle (e.g., [15, 22, 30]). A number of researchers have successfully used Markov
 31 regime-switching models to fit the dynamics of the short-term interest rate (see,
 among others, [2, 23, 28, 29]).¹ These results have motivated the recent studies of
 the impact of regime shifts on the entire yield curve using dynamic term structure
 33 models. A common approach, as in Naik and Lee [35], Boudoukh *et al.* [6], Evans [20]
 and Bansal and Zhou [3], is to incorporate Markov-switching into the stochastic

¹The expectation theory is usually invoked to relate long-term interest rates to the short rate in
 this literature, such as in Ang and Bekaert [2].

2 *S. Wu & Y. Zeng*

1 processes of the pricing kernel and/or state variables.² The regime-dependence intro-
2 duced by these studies implies richer dynamic behavior of the market price of risk
3 and therefore offers greater econometric flexibility for the term structure models to
4 simultaneously account for the time series and cross-sectional properties of interest
5 rates. However, as pointed out by Dai and Singleton [12], the risk of regime shifts is
6 not priced in these models, and hence it does not contribute independently to bond
7 risk premiums.

8 Without pricing the risk of regime shifts, the previous studies have essentially
9 treated the regime shifts as an idiosyncratic risk that can be diversified away by
10 bond investors. However, most of these studies (see, for example, [3]) have also
11 shown that regimes are intimately related to the business cycle, suggesting a close
12 link between the regime shift and aggregate uncertainties.³ Hence it is most likely
13 that the Markov regime shifts represent a systematic risk which should be priced
14 in the term structure models.

15 The main objective of current paper is to extend this strand of literature by
16 developing a dynamic term structure model under the systematic risk of regime
17 shifts in a general equilibrium setting similar to Cox, Ingersoll and Ross [9, 10]. Our
18 model implies that bond risk premiums include two components under regime shifts.
19 One is a regime-dependent risk premium due to diffusion risk as in the previous
20 studies. This risk premium has added econometric flexibilities, as demonstrated in
21 the previous empirical studies, relative to those in single-regime models because of
22 the Markov-shift of the underlying parameters. The other is a regime-switching risk
23 premium that depends on the covariations between the discreet changes in marginal
24 utility and bond prices across different regimes. Therefore the model shows that
25 there is an additional source of time-variation in bond risk premiums in the presence
26 of regime shifts. This new component of the term premiums is associated with the
27 systematic risk of periodic shifts in bond prices (or interest rates) due to regime
28 changes. Given the empirical evidence from the previous studies that the yield curve
29 exhibits significantly different properties across regimes, the model implies that the
30 regime-switching risk is likely to be an important factor that affects bond returns.

31 We further obtain a closed-form solution of the term structure of interest rates
32 under an affine-type model using the log-linear approximation similar to that in
33 Bansal and Zhou [3]. The model is estimated by Efficient Method of Moments
34 (EMM) developed in Bansal *et al.* [4] and Gallant and Tauchen [24, 26]. We use the
35 monthly data on 6-month treasury bill and 5-year treasury bond from 1964 to 2000
36 in the estimation. We find that the market price of regime-switching risk is highly
37 significant and affects mostly the long-end of the yield curve. The regime-switching

²Other studies of the term structure of interest rates under hidden Markov chains include Bielecki and Rutkowski [5], Elliott and Mamon [18] among others. For regime-shifts in option pricing, see Di Masi *et al.* [14]. For regime-shifts in irreversible investment, see Guo *et al.* [27].

³Following Bansal and Zhou [3], we plot in Fig. 2 the implied regimes by the term structure model in Sec. 3 below and the business cycle expansions and recessions identified by NBER. The figure clearly shows that the regimes are related to the fluctuations of the aggregate economy.

1 risk, as expected, accounts for a significant portion of the term premiums for long-
 2 term bond.

3 In a closely related study, Dai, Singleton and Yang [13] develop and estimate
 4 a multi-factor dynamic term structure model with regime-shift risk based on no-
 5 arbitrage condition. Their results also strongly reject the null hypothesis that the
 6 regime-shift risk is not priced in the US treasury bond yields. They show that the
 7 model with priced regime-switching risk can capture some salient features of term
 8 structure of interest rates. The focus of this paper is to derive the term structure
 9 model from a general equilibrium setting that allows us to further elucidates the
 10 economic nature of the regime-switching risk, which is the main difference between
 11 the current paper and Dai *et al.* [13]. Another difference is our model is specified in
 12 continuous time and the model allows the factor loadings in the term structure to
 13 be regime-dependent, while the model in Dai *et al.* [13] is in discrete time and the
 14 regime shifts have no impact on the factor loadings.⁴

15 The rest of the paper is organized as follows. Section 2 specifies the underlying
 16 economy and lays out the asset pricing model under the systematic regime-switching
 17 risk. Section 3 obtains the closed-form solution for the term structure of interest
 18 rates. Section 4 discusses the empirical results from EMM estimation and Sec. 5
 19 concludes.

2. The Model

21 2.1. *The underlying economy*

22 Consider an economy with a single good and a large number of infinitely lived and
 23 identical agents similarly to that in Cox, Ingersoll and Ross [9, 10] (henceforth CIR).
 24 We first describe the state variables and investment opportunities for the economy
 25 as well as the representative agent's objective function below.

2.1.1. *State variables*

27 We assume that the economy is driven by two state variables.⁵ One of the state
 28 variable $x(t)$ has a continuous path and is determined by the stochastic differential
 29 equation

$$dx = \mu_x dt + \sigma_x dB_t, \quad (2.1)$$

where the drift term μ_x and the diffusion term σ_x are in general time-varying and
 regime-dependent. The other state variable is a continuous-time Markov chain $s(t)$

⁴The latter feature of the model in Dai *et al.* [13] allow them to obtain the exact solution to the
 term structure of interest rates without resorting to the log-linear approximation. Their model also
 allows the regime-switching probabilities to be state-dependent. While the transition intensities
 across regimes are assumed to be constant in the estimated term structure model of the current
 paper, our main theoretical results do not depend on this assumption.

⁵It is straight forward to extend the model to include more state variables. We keep the model as
 simple as possible to ease exposition.

4 *S. Wu & Y. Zeng*

taking on values of $1, 2, \dots, N$, representing N distinct regimes. Following Landen [32], we make use of the marked point process to obtain a convenient representation of $s(t)$. In particular, the mark space E , which contains all possible regime shifts, is defined as

$$E = \{(i, j) : i \in \{1, \dots, N\}, j \in \{1, \dots, N\}, i \neq j\},$$

1 with σ -algebra $\mathcal{E} = 2^E$. Denote $z = (i, j)$ as a generic point in E , representing a
 3 regime shift from state i to j . A marked point process, $m(t, \cdot)$ is uniquely characterized by its stochastic intensity kernel,⁶ which can be defined as

$$\gamma_m(dt, dz) = h(z, x(t-))\mathbf{I}\{s(t-) = i\}\epsilon_z(dz)dt, \quad (2.2)$$

5 where $h(z, x(t-))$ is the regime-shift (from regime i to j) intensity at $z = (i, j)$
 6 conditional on $x(t-)$, $\mathbf{I}\{s(t-) = i\}$ is an indicator function of regime at time $t-$,
 7 and $\epsilon_z(A)$ is the Dirac measure for A , a subset of E , at point z (defined by $\epsilon_z(A) = 1$
 9 if $z \in A$ and 0, otherwise). Heuristically, for $z = (i, j)$, $\gamma_m(dt, dz)$ can be thought of
 as the conditional probability of shifting from Regime i to Regime j during $[t, t+dt)$
 given $x(t-)$ and $s(t-) = i$. Note that $\gamma_m(dt, dz)$ is in general state dependent.

11 Let A be a subset of E . Then $m(t, A)$ counts the cumulative number of regime
 shifts that belong to A during $(0, t]$. $m(t, A)$ has its compensator, $\gamma_m(t, A)$, given by

$$\gamma_m(t, A) = \int_0^t \int_A h(z, x(\tau-))\mathbf{I}\{s(\tau-) = i\}\epsilon_z(dz)d\tau. \quad (2.3)$$

This simply implies that $m(t, A) - \gamma_m(t, A)$ is a martingale.

15 Using the above notations, the evolution of the regime $s(t)$ can be conveniently
 represented as

$$17 \quad ds = \int_E \zeta(z)m(dt, dz), \quad (2.4)$$

with the compensator given by

$$19 \quad \gamma_s(t)dt = \int_E \zeta(z)\gamma_m(dt, dz), \text{ where } \zeta(z) = \zeta((i, j)) = j - i. \quad (2.5)$$

21 For example, if there is a regime shift from i to j occurs at time t , Eq. (2.4) then
 implies $s_t = (j - i) + s_{t-} = j$. Note that \int_E is equivalent to the double summation
 $\sum_{j \neq i} \sum_i$.

23 2.1.2. *Investment opportunities*

Without loss of generality, we assume that the output is produced by a single
 technology which depends on both state variables $x(t)$ and $s(t)$ as described by the

⁶See Last and Brandt [33] for detailed discussion of marked point process, stochastic intensity kernel and related results.

1 following stochastic differential equation

$$dy = y\mu_y dt + y\sigma_y dB_t + \int_E y\delta_y(z)m(dt, dz), \quad (2.6)$$

3 where both μ_y and σ_y can be functions of $x(t-)$ and $s(t-)$. And $\delta_y(z)$ is the discrete percentage change in y due to a regime shift, i.e., $\delta_y(z) = \frac{y(t,s(t)) - y(t-,s(t-))}{y(t-,s(t-))}$.
 5 Therefore we assume that regime shifts not only affect the drift μ_y and the volatility σ_y , but also directly result in discontinuous changes in the output process as the
 7 economy shifts from one regime to another.

As in CIR model, we assume that there is a competitive market for instantaneous borrowing and lending at the short-term interest rate $r(t)$. There is also a competitive market for default-free pure discount bonds whose prices are given by

$$dF = F\mu_F dt + F\sigma_F dB_t + \int_E F\delta_F(z)m(dt, dz), \quad (2.7)$$

11 where μ_F , σ_F and $\delta_F(z)$ are to be determined by the equilibrium conditions. And $\delta_F(z)$ is the discrete percentage change in the bond prices due to a regime shift, i.e., $\delta_F(z) = \frac{F(t,s(t)) - F(t-,s(t-))}{F(t-,s(t-))}$. Hence, unlike the conventional models, we allow
 13 regime shifts not only to affect the conditional distribution of bond prices (μ_F, σ_F) , but also to lead to discontinuous jumps in the prices when regime shifts occur.

Note that (2.7) can be alternatively written as

$$dF = F\mu_F dt + \int_E F\delta_F(z)\gamma_m(dt, dz) + F\sigma_F dB_t + \int_E F\delta_F(z)[m(dt, dz) - \gamma_m(dt, dz)]. \quad (2.8)$$

17 The last two terms in the above equation are martingales. Therefore (2.8) implies that the instantaneous expected bond return is

$$E_{t-} \left(\frac{dF}{F} \right) = \mu_F dt + \int_E \delta_F(z)\gamma_m(dt, dz), \quad (2.9)$$

19 where the first term is the regime-dependent expected bond return due to diffusion, and the last term is an additional component in the bond return due to discrete
 21 regime shifts.

23 2.1.3. The representative agent's objective function

Given the initial wealth w_0 , a representative agent seeks to maximize her expected lifetime utility given by

$$E_0 \left[\int_0^\infty e^{-\rho t} U(c(t)) dt \right], \quad (2.10)$$

27 where $c(t)$ is the flow of consumption and $U(\cdot)$ is the instantaneous utility function. As usual, $U(\cdot)$ is assumed to be strictly concave, increasing and twice differentiable
 29 with $U(0) = 0$ and $U'(0) = \infty$.

6 *S. Wu & Y. Zeng*

1 At each instant, the representative agent allocates her wealth among investment
 2 in the physical production, the discount bonds, the risk-free borrowing and lend-
 3 ing and consumption. We assume that both physical investment and trading of the
 4 financial assets take place in continuous time without borrowing constraint, trans-
 5 action cost and all other forms of market friction. The agent's budget constraint is
 6 therefore given by

$$7 \quad dw = w\mu_w dt + w\sigma_w dB_t + \int_E w\delta_w(z)m(dt, dz), \quad (2.11)$$

8 where

$$9 \quad w\mu_w = w[\phi_1(\mu_y - r) + \phi_2(\mu_F - r) + r] - c \quad (2.12)$$

$$10 \quad w\sigma_w = w[\phi_1\sigma_y + \phi_2\sigma_F] \quad (2.13)$$

$$11 \quad w\delta_w(z) = w[\phi_1\delta_y(z) + \phi_2\delta_F(z)]. \quad (2.14)$$

12 In the above equations, $w(t)$ is the agent's wealth at time t , ϕ_1 is the proportion
 13 of her wealth invested in the physical production, ϕ_2 is the proportion of her wealth
 14 invested in the discount bonds, and $c(t)$ is her flow of consumption. As we can clearly
 15 see from these equations, regime shifts have two effects on the investor's wealth
 16 process. On one hand, they affect the drift and diffusion terms of the investor's
 17 wealth (both μ_w and σ_w are regime-dependent). On the other hand, they also
 18 create discontinuous changes in the wealth process when the economy shifts between
 19 distinct regimes. This is a more general setup than the conventional models where
 20 regime shifts are assumed to only have the first effect on the investor's wealth.

17 2.2. The equilibrium bond returns

18 In this section we state the main asset pricing results for the economy subject to
 19 the systematic regime-switching risk. The detailed derivations are provided in the
 20 appendices.

21 Let $J(w(t), s(t), x(t)) = \sup_{\{c, \phi_1, \phi_2\}} E_t[\int_t^\infty e^{-\rho(\tau-t)} U(c(\tau)) d\tau]$, be the indirect
 22 utility function. We use the following notations in our discussions below: $J_w =$
 23 $\frac{\partial J}{\partial w}$, $J_x = \frac{\partial J}{\partial x}$, $J_{ww} = \frac{\partial^2 J}{\partial w^2}$ and $J_{wx} = \frac{\partial^2 J}{\partial w \partial x}$. We also use w^c to represent the
 24 continuous part of the wealth process. As in CIR, we denote $\text{Var}(w^c) = (w\sigma_w)^2$,
 25 $\text{Var}(x) = \sigma_x^2$, and $\text{Cov}(w^c, x) = (w\sigma_w)\sigma_x$. To further simplify notations, we define
 26 $\Delta_s f = f(s(t)) - f(s(t-))$ for any function $f(\cdot)$ that depends on $s(t)$. $\Delta_s f$ is therefore
 27 the difference in $f(\cdot)$ due to a regime shift at time t .

28 The following two propositions give the equilibrium instantaneous short-term
 29 interest rate and the expected excess rate of return on a bond respectively.

Proposition 2.1. *The equilibrium short-term interest rate is given by*

$$30 \quad r = \mu_y^* - \left(-\frac{J_{ww}}{J_w} \right) \frac{\text{Var}(w^c)}{w} - \left(-\frac{J_{wx}}{J_w} \right) \frac{\text{Cov}(w^c, x)}{w} \\
 31 \quad - \int_E \left(-\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w} \gamma_m(dz), \quad (2.15)$$

1 where

$$\mu_y^* = \mu_y + \int_E \delta_y(z) \gamma_m(dz), \quad (2.16)$$

3 and

$$\gamma_m(dz) = h(z, x(t-)) \mathbf{I}\{s(t-) = i\} \epsilon_z(dz). \quad (2.17)$$

Note that $\mu_y^* dt$ is the instantaneous expected rate of return on the production technology $E_{t-} \left(\frac{dy(t)}{y(t-)} \right)$. And $\frac{\Delta_s w}{w}$ and $\frac{\Delta_s J_w}{J_w}$ in (2.15) are the discrete percentage changes in w and J_w respectively due to a regime shift, i.e., $\frac{\Delta_s w}{w} = \delta_w(z) = \phi_1 \delta_y(z) + \phi_2 \delta_F(z)$, and

$$\frac{\Delta_s J_w}{J_w} = \frac{J_w(w(1 + \delta_w(z)), s + \zeta(z), x) - J_w(w, s, x)}{J_w(w, s, x)}.$$

Proposition 2.1 implies that the instantaneous short-term interest rate $r(t)$ is a function of both state variables $x(t)$ and $s(t)$ because μ_y^* , $\text{Var}(w^c)$, $\text{Cov}(w^c, x)$ and the marginal utility J_w are regime-dependent. If the regime shifts is not a systematic risk as assumed in the previous literature, $\frac{\Delta_s J_w}{J_w}$ would be equal to zero. Otherwise the short-term interest rate will also be affected by this term. Since the indirect utility function $J(w, s, x)$ is concave in w , if it is also separable in w and s (as in the case of log utility function), we will have

$$\left(-\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w} > 0.$$

5 Therefore the systematic regime-switching risk tends to lower the equilibrium short-
 7 term interest rate, as the local risk-free borrowing and lending opportunity offers
 9 a hedge against such risk. This result is similar to the impact of a systematic
 11 jump risk on the interest rate as shown in Ahn and Thompson [1]. Nevertheless,
 $\left(-\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w}$ is not always positive in general, because J_w also depends on s .
 Therefore the regime-switching risk has a different effect than that of jump risk
 even though they both lead to discontinuous changes in the marginal utility and
 wealth.

Proposition 2.2. Let $\mu_F^* dt = E_{t-} \left(\frac{dF(t)}{F(t-)} \right)$, be the instantaneous expected rate of return of a discount bond. In equilibrium, we have

$$\begin{aligned} \mu_F^* - r &= \left[\left(-\frac{J_{ww}}{J_w} \right) \text{Var}(w^c) + \left(-\frac{J_{wx}}{J_w} \right) \text{Cov}(w^c, x) \right] \frac{F_w}{F} \\ &+ \left[\left(-\frac{J_{ww}}{J_w} \right) \text{Cov}(w^c, x) + \left(-\frac{J_{wx}}{J_w} \right) \text{Var}(x) \right] \frac{F_x}{F} \\ &+ \int_E \left(-\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s F}{F} h(z, x) \mathbf{I}\{s = i\} \epsilon_z(dz), \end{aligned} \quad (2.18)$$

13 where $F_w = \frac{\partial F}{\partial w}$, $F_x = \frac{\partial F}{\partial x}$, and $\frac{\Delta_s F}{F} = \delta_F(z)$ is the discrete percentage change in the bond price due to a regime shift.

8 *S. Wu & Y. Zeng*

1 We can clearly see from the above proposition that, in a single regime setting,
 2 $\frac{\Delta_s F}{F} = 0$, $\frac{\Delta_s J_w}{J_w} = 0$ and $\frac{\Delta_s w}{w} = 0$, and (2.18) is reduced to the standard result
 3 in CIR. On the other hand, in the term structure models with regime shifts such
 4 as those in Bansal and Zhou [3] and Evans [20] among others, $\text{Var}(w^c)$, $\text{Var}(x)$
 5 and $\text{Cov}(w^c, x)$ are assumed to be regime-dependent. Therefore these models have
 6 additional econometric flexibility due to the regime-dependent parameters and are
 7 shown to have better empirical performance than single-regime models. One implicit
 8 assumption maintained by these models, however, is that the regime shift is not a
 9 systematic risk to investors, i.e., $\frac{\Delta_s J_w}{J_w} = 0$.

10 To further illustrate the role played by $\frac{\Delta_s J_w}{J_w}$ in determining bond returns, let's
 11 assume that $J_{wx} = 0$ and $F_w = 0$.⁷ We can obtain from (2.18) that

$$\mu_F^* - r = -\frac{J_{ww}}{J_w} w \sigma_w \frac{\sigma_x F_x}{F} - \int_E \frac{\Delta_s J_w}{J_w} \frac{\Delta_s F}{F} h(z, x) \mathbf{I}\{s = i\} \epsilon_z(dz). \quad (2.19)$$

13 The first term in (2.19) is the instantaneous diffusion risk premium. $\frac{\sigma_x F_x}{F}$ is the
 14 volatility of the bond return due to diffusions in $x(t)$. $(-\frac{J_{ww}}{J_w}) w \sigma_w$ measures the
 15 excess rate of return per unit of such volatility, commonly referred to as the market
 16 price of risk in the literature. The second term is the instantaneous regime-switching
 17 risk premium, where $\frac{\Delta_s F}{F} h(z, x)$ is the expected discrete percentage change in the
 18 bond price due to a shift from regime i to j , and $(-\frac{\Delta_s J_w}{J_w})$ measures the excess
 19 bond return per unit of such expected changes. Hence $(-\frac{\Delta_s J_w}{J_w})$ can be analogously
 20 defined as the market price of regime-switching risk. And we can see from (2.19),
 21 unless J_w (the marginal utility) is independent of regime $s(t)$, equilibrium bond
 22 prices will reflect the risk of regime shifts.

23 Using Ito's formula, the diffusion term of the bond price is given by $F \sigma_F =$
 24 $(w \sigma_w) F_w + \sigma_x F_x$. Hence more insight can be obtained by rewriting (2.18) as

$$\mu_F^* - r = -\text{Cov}\left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F}\right) - \int_E \frac{\Delta_s J_w}{J_w} \frac{\Delta_s F}{F} h(z, x) \mathbf{I}\{s = i\} \epsilon_z(dz), \quad (2.20)$$

where

$$\text{Cov}\left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F}\right) = \left(\frac{J_{ww} w \sigma_w + J_{wx} \sigma_x}{J_w}\right) \frac{\sigma_F F}{F}.$$

27 (2.20) implies that the expected excess bond return includes two components
 28 under regime shifts. The first term depends on the covariance of the continuous
 29 part of the bond return and continuous part of the rate of change in the marginal
 30 utility of wealth. The second term depends on the covariation between the discrete
 31 percentage change in the marginal utility and the discrete percentage change in the
 32 bond price under regime shifts. The higher the covariation, the greater the payoff
 33 the bond provides when marginal utility is higher. Hence agents are willing to accept
 a lower expected rate of return on the asset.

⁷For example, both $J_{wx} = 0$ and $F_w = 0$ hold under log utility function for discount bonds as shown in CIR.

1 In the previous regime-switching term structure models, greater flexibilities are
 2 obtained in fitting the time-varying excess bond returns by forcing $\text{Cov}\left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F}\right)$
 3 regime-dependent. Equation (2.20) shows that allowing investors to price the
 4 regime-switching risk (i.e., $\frac{\Delta_s J_w}{J_w} \neq 0$) introduces an additional source of time-
 5 variation in the expected excess bond returns. This new component in the bond
 6 returns is associated with the potentially large shifts in bond prices across different
 7 regimes $\frac{\Delta_s F}{F}$. It also depends on the regime-shift intensity $h(z, x)$ as well as the
 8 market price of regime-switching risk $-\frac{\Delta_s J_w}{J_w}$. Holding $\frac{\Delta_s F}{F}$ and $-\frac{\Delta_s J_w}{J_w}$ constant,
 9 the higher the regime-shift intensity $h(z, x)$, the larger the risk premium.⁸ On the
 10 other hand, given $(-\frac{\Delta_s J_w}{J_w})$ and $h(z, x)$, the regime-switching risk premium depends
 11 on the magnitude $\frac{\Delta_s F}{F}$. The bigger the difference in the bond price across differ-
 12 ent regimes, the more important the risk premium due to regime shifts. Empirical
 13 results from the previous studies (e.g., [3]) imply sizeable $\frac{\Delta_s F}{F}$, hence suggesting
 14 that the regime-switching risk premium is likely to be an important component of
 15 bond returns.

3. The Term Structure of Interest Rates

17 In this section we obtain a closed form solution for the term structure of interest
 18 rates. To do so, we assume that $U(c) = \log(c)$. The prices of default-free pure
 19 discount bonds are then given by the following proposition (see the Appendix C for
 proof).

Proposition 3.1. *The price at time t of a default-free pure discount bond $F(t, x(t), s(t), T)$ that matures at time T satisfies the following system of partial differential equations*

$$F_t + (\mu_x - \sigma_y \sigma_x) F_x + \frac{1}{2} \sigma_x^2 F_{xx} + \int_E (1 - \lambda_s(z)) \Delta_s F h(z, x) \mathbf{I}\{s = i\} \epsilon_z(dz) = rF \quad (3.1)$$

21 *with the boundary condition: $f(T, x, s, T) = 1$ for all x and s .*

23 In the above equation, $F_t = \frac{\partial F}{\partial t}$, $F_x = \frac{\partial F}{\partial x}$, $F_{xx} = \frac{\partial^2 F}{\partial x^2}$, $\Delta_s F = F(t, x(t), s(t-) + \zeta(z), T) - F(t-, x(t-), s(t-), T)$, and $\lambda_s(z) = \frac{\delta_y(z)}{1 + \delta_y(z)}$. Note that (3.1) holds for each regime of $s(t)$, it therefore defines a system of N partial differential equations if there are N distinct regimes. Moreover, under the log utility function, the equilibrium

⁸Boudoukh *et al.* [6] found that economic turning points are usually characterized by highly volatile and strongly time-varying term premium.

10 *S. Wu & Y. Zeng*

1 short-term interest rate can be obtained from Proposition 2.1 as

$$r = \mu_y - \sigma_y^2 + \int_E \lambda_s(z) h(z, x) \mathbf{I}\{s = i\} \epsilon_z(dz). \quad (3.2)$$

In general the system (3.1) does not admit a closed form solution to the bond price. Hence we consider the following affine specification which is known to offer a tractable model of the term structure of interest rate.⁹ In particular, we assume

$$\mu_x = a_0(s) + a_1(s)x, \quad (3.3)$$

$$\sigma_x = \sqrt{\sigma(s)}x, \quad (3.4)$$

$$h(z, x) = e^{\eta_s(z)}, \quad (3.5)$$

$$\sigma_y = \theta_x(s) \sqrt{\sigma(s)}x, \quad (3.6)$$

$$\mu_y = x + \theta_x^2(s) \sigma(s)x - \int_E \lambda_s(z) \gamma_m(dz), \quad (3.7)$$

$$\lambda_s(z) = 1 - e^{\theta_s(z)}, \quad (3.8)$$

3 where $\gamma_m(dz) = h(z, x) \mathbf{I}\{s = i\} \epsilon_z(dz)$.

5 Under (3.3) and (3.4), the state variable $x(t)$ follows a conventional mean-reverting square-root process with regime-dependent drift and diffusion terms

$$dx = (a_0(s) + a_1(s)x) dt + \sqrt{\sigma(s)}x dB_t. \quad (3.9)$$

7 Equation (3.6) and (3.7) are the assumptions about the drift term and the diffusion term of the physical production process $y(t)$. Under the log utility function, these
9 imply that the market price of the diffusion risk in equilibrium is given by

$$\lambda_x(s) = \theta_x(s) \sqrt{\sigma(s)}x. \quad (3.10)$$

11 Equation (3.5) assumes that the Markov chain $s(t)$ has constant transition probabilities given by $e^{\eta_s(z)}$ for simplicity. Equation (3.8) parameterizes the market price of
13 the regime-switching risk $\lambda_s(z)$ as $1 - e^{\theta_s(z)}$. Moreover (3.2), (3.6) and (3.7) together imply that $r(t) = x(t)$. Therefore we can write

$$15 \quad dr = \kappa(s)(\bar{r}(s) - r) dt + \sqrt{\sigma(s)}r dB_t, \quad (3.11)$$

where $\kappa(s) = -a_1(s)$, $\bar{r}(s) = \frac{a_0(s)}{-a_1(s)}$.

17 Appendix D shows that, using a log-linear approximation similar to that in Bansal and Zhou [3], the term structure of interest rates can be solved as follows

Proposition 3.2. *Under assumptions (3.3)–(3.8), the price at time t of a default-free pure discount bond with maturity τ is given by $F(t, \tau) = e^{A(\tau, s(t)) + B(\tau, s(t))r(t)}$*

⁹Duffie and Kan [16] and Dai and Singleton [11] offer detailed discussions of affine term structure models under diffusions. Bansal and Zhou [3], Evans [20] and Landen [32] also use affine specifications in their regime-switching models.

and the τ -period interest rate is given by $R(t, \tau) = -\frac{A(\tau, s(t))}{\tau} - \frac{B(\tau, s(t))r(t)}{\tau}$, where $A(\tau, s)$ and $B(\tau, s)$ are determined by the following system of differential equations

$$-\frac{\partial B(\tau, s)}{\partial \tau} + [a_1(s) - \theta_x(s)\sigma(s)]B(\tau, s) + \frac{1}{2}\sigma(s)B^2(\tau, s) + \int_E (e^{\Delta_s A} \Delta_s B) e^{\eta_s(z) + \theta_s(z)} \mathbf{1}(s = i) \epsilon_z(dz) = 1, \quad (3.12)$$

1 and

$$-\frac{\partial A(\tau, s)}{\partial \tau} + a_0(s)B(\tau, s) + \int_E (e^{\Delta_s A} - 1) e^{\eta_s(z) + \theta_s(z)} \mathbf{1}(s = i) \epsilon_z(dz) = 0, \quad (3.13)$$

3 with boundary conditions $A(0, s) = 0$ and $B(0, s) = 0$, where $\Delta_s A = A(\tau, s + \zeta(z)) - A(\tau, s)$ and $\Delta_s B = B(\tau, s + \zeta(z)) - B(\tau, s)$.

5 Proposition 3.2 nests the models in Bansal and Zhou [3] and Landen [32]. With-
 7 out using the log-linear approximation, Landen [32] only considers models where
 9 $\Delta_s B = 0$ and is silent on the market prices of risk $\theta_x(s)$ and $\theta_s(z)$. In Bansal and
 11 Zhou [3], the risk of regime shifts is not priced, namely $\theta_s(z)$ is assumed to be
 13 zero. Admittedly, the model in Proposition 3.2 still makes some strong assumptions
 such as constant regime-switching probabilities, even though the theoretical results
 in the previous section do not depend on these assumptions. In the next section,
 we provide some empirical results regarding the risk of regime shifts in the term
 structure of interest rates based on the model above.

4. Empirical Results

15 4.1. Data and summary statistics

17 The data used in this study are monthly interest rates from June 1964 to November
 19 2000 obtained from the Center for Research in Security Prices (CRSP). There are
 21 eight interest rates with maturities ranging from one month to five years. Table 1
 contains their summary statistics. We can see that the yield curve is on average
 upward-sloping and the large skewness and kurtosis suggest significant departure
 from Gaussian distribution. As Timmermann [39] shows, Markov switching models
 can generate such large skewness and kurtosis.

23 We also report in Table 2 the results from regressions of excess bond returns on
 25 forward interest rates and a business cycle dummy variable. We use NBER dates
 27 of business cycles to distinguish between expansions and recessions. As in the pre-
 29 vious literature we find significant coefficients on forward rates in the regression,
 31 suggesting that forward rates contain information about the state variables driving
 the interest rates. More importantly, the coefficients on the business cycle dummy
 variable are all negative and significant, suggesting important regime-dependent
 property of the bond returns. Note that the sign of the coefficient on the busi-
 ness cycle dummy variable is consistent with the counter-cyclical behavior of risk
 premiums as documented in Fama and French [21].

12 *S. Wu & Y. Zeng*

Table 1. Summary statistics of the term structure of interest rates 1964–2000 (the first column indicates the maturities of the interest rates).

	Mean	Std	Corr	Skew	Kurt	Max	Min
1-month	0.05608	0.025452	1	1.28697	5.17923	0.161376	0.013176
3-month	0.060275	0.026706	0.985712	1.26099	4.91011	0.160332	0.017112
6-month	0.062443	0.026954	0.978383	1.21604	4.69365	0.165168	0.017808
1-year	0.06455	0.026314	0.964593	1.07663	4.21845	0.15812	0.01956
2-year	0.066561	0.025774	0.938704	1.01143	3.93613	0.15639	0.02368
3-year	0.068159	0.025069	0.916046	0.998881	3.84814	0.15557	0.0296
4-year	0.069354	0.024795	0.897531	0.965216	3.74017	0.15824	0.03118
5-year	0.069981	0.024589	0.883406	0.921516	3.57469	0.15001	0.03346

Note: mean = average interest rate; std = standard deviation; corr = correlation coefficient between long rates and the 1-month interest rate; skew = skewness; kurt = kurtosis; max = maximum interest rate; min = minimum interest rate.

Table 2. Using forward rates to predict excess bond returns.

	3-month	6-month	1-year	2-year	3-year	4-year	5-year
Constant	0.0030 (0.0022)	0.0116 (0.0054)	0.0188 (0.0124)	0.0224 (0.0227)	0.0236 (0.0321)	0.0187 (0.0321)	0.0161 (0.0455)
Bus-cycle	-0.0074 (0.0021)	-0.0167 (0.0050)	-0.0313 (0.0111)	-0.0495 (0.0199)	-0.0619 (0.0272)	-0.0694 (0.0272)	-0.0773 (0.0371)
f1	-0.2145 (0.0979)	-0.5015 (0.2362)	-0.1386 (0.5121)	-0.8860 (0.8805)	-1.4536 (1.1746)	-1.9372 (1.1746)	-2.2324 (1.7405)
f6	0.1363 (0.1039)	0.3895 (0.1963)	-0.2358 (0.4121)	-0.0988 (0.7022)	-0.0620 (0.9602)	-0.3291 (0.9602)	-0.6753 (1.4695)
f60	0.1935 (0.0694)	0.2128 (0.1458)	0.5717 (0.3320)	1.2403 (0.6012)	1.8126 (0.8451)	2.6277 (0.8451)	3.3306 (1.2642)

Note: The first row indicates maturities of the bonds. The first column includes the explanatory variables in the regression. Bus-cycle is a business cycle dummy variable according to NBER business dates. Bus-cycle = 1: expansion; Bus-cycle = 0: recession. f1 = 1-month rate, f6 = 6-month forward rate, f60 = 5-year forward rate. Numbers in parentheses are Newy–West standard errors.

1 **4.2. Econometric methodology**

2 We use Efficient Method of Moments¹⁰ (EMM) to estimate the term structure model
3 in Proposition 3.2. We assume that there are two distinct regimes ($N = 2$) for $s(t)$.
4 Therefore (3.12) and (3.13) define a system of four differential equations that must
5 be solved simultaneously. The model has a total of 12 parameters. As in Bansal and
6 Zhou [3], we fit the model to the data on the 6-month and the 5-year rates.

7 Under EMM procedure, the empirical conditional density of the observed
interest rates is first estimated by an auxiliary model that is a close approximation

¹⁰Bansal *et al.* [4] and Gallant and Tauchen [24, 26] contain detailed discussion of EMM. Bansal and Zhou [3] is an excellent example of applying EMM to estimate the term structure model under regime shifts. Dai and Singleton [11] also provides extensive discussions of estimating affine term structure models using EMM procedure.

1 to the true data generating process. Gallant and Tauchen [26] suggests a
 2 semi-nonparametric (SNP) series expansion as a convenient general purpose aux-
 3 iliary model. As pointed out by Bansal and Zhou [3], one advantage of using the
 4 semi-nonparametric specification for the auxiliary model is that it can asymptot-
 5 ically converge to any smooth distributions [25], including the density of Markov
 6 regime-switching models. The dimension of this auxiliary model can be selected
 7 by, for example, the Schwarz's Bayesian Information Criterion (BIC)¹¹. The score
 8 function of the auxiliary model are then used to generate moment conditions for
 9 computing a chi-square criterion function, which can be evaluated through simula-
 10 tions given the term structure model under consideration. A nonlinear optimizer is
 11 used to find the parameter setting that minimizes the criterion function. Gallant
 12 and Tauchen [24] shows that such estimation procedure yields fully efficient estima-
 13 tors if the score function of the auxiliary model encompasses the score functions of
 the model under consideration.

15 4.3. Results

16 Tables 3 and 4 contain the results from SNP estimation. In searching for the pre-
 17 ferred specification, we follow closely Bansal and Zhou [3] mainly because we use
 18 the same interest rate data (with longer sample period). Different choices of SNP
 19 density and their corresponding BIC values are reported in Table 3. Consistent with
 20 Bansal and Zhou [3], we find that the SNP specification with one lag ($L_\mu = 1$) in the
 21 VAR-based conditional mean, five lags in ARCH term ($L_r = 5$) and a polynomial
 22 of order four ($K_z = 4$) in the standardized residual z has the overall best fit based
 23 on BIC. Estimates of the coefficients in the preferred SNP density are reported in
 Table 4.

24 Given the SNP density, the parameters of the term structure model can be esti-
 25 mated via EMM through simulations¹². We estimate three versions of CIR model.
 26 Model One is the standard one-factor CIR model. Model 2 is a one-factor CIR
 27 model with regime shifts, but the risk of regime shifts is not priced. We allow the
 28 parameters in the diffusion process of the instantaneous short-term interest rate $r(t)$
 29 to be regime-dependent. We also assume different market prices of the diffusion risk
 30 across regimes. Model 3 is the one-factor CIR developed in Proposition 3.2 where
 31 the risk of regime shifts is priced. The results are reported in Table 5.¹³

¹¹As for model selection for regime switching models (or general Hidden Markov models), Scott [37] gave an excellent review on limitations of various criteria, including BIC and AIC.

¹²Eular–Maruyama scheme is used to simulate sample paths of Models 1, 2 and 3, which do not satisfy the Lipschitz condition and the usual convergence and consistency results for discrete-time approximation fails. However, recent results in Mao, Yuan and Yin [34] guarantee the consistency of Euler–Maruyama approximation.

¹³We did not estimate a multi-factor model mainly because the main purpose of this section is not to obtain a good empirical fitting, but to highlight the potential impact of the systematic risk of regime shifts on the term structure of interest rates. Bansal and Zhou [3] and Dai *et al.* [13] provide excellent estimation results of multi-factor term structure models under regime shifts.

Table 3. SNP specifications.

L_μ	L_r	L_p	K_z	I_z	K_x	I_x	l_θ	$s_n(\hat{\theta})$	BIC
1	0	1	0	0	0	0	9	-.82698	-.76277
2	0	1	0	0	0	0	13	-.85153	-.75897
3	0	1	0	0	0	0	17	-.85784	-.73656
1	1	1	0	0	0	0	11	-1.02082	-.94234
1	2	1	0	0	0	0	13	-1.05923	-.96648
1	3	1	0	0	0	0	15	-1.10226	-.99525
1	4	1	0	0	0	0	17	-1.15356	-1.03228
1	5	1	0	0	0	0	19	-1.17153	-1.03598
1	6	1	0	0	0	0	21	-1.17394	-1.02412
1	7	1	0	0	0	0	23	-1.18247	-1.01839
1	5	1	2	1	0	0	23	-1.18124	-1.01716
1	5	1	2	0	0	0	24	-1.22077	-1.04956
1	5	1	3	2	0	0	25	-1.18248	-1.00413
1	5	1	3	1	0	0	26	-1.22941	-1.04393
1	5	1	3	0	0	0	28	-1.23006	-1.03030
1	5	1	4	3	0	0	27	-1.22163	-1.02901
1	5	1	4	2	0	0	28	-1.25705	-1.05729
1	5	1	4	1	0	0	30	-1.26133	-1.04703
1	5	1	4	0	0	0	33	-1.28814	-1.05272
1	5	1	5	4	0	0	29	-1.22625	-1.01936
1	5	1	5	3	0	0	30	-1.25837	-1.04435
1	5	1	5	2	0	0	32	-1.26270	-1.03441
1	5	1	5	1	0	0	35	-1.29353	-1.04383
1	5	1	5	0	0	0	39	-1.29058	-1.01235
1	5	1	4	2	1	0	48	-1.32073	-.97829
1	5	1	4	2	2	1	68	-1.36002	-.87490
1	5	1	4	2	2	0	78	-1.37679	-.82033

Note: L_μ is the number of lags in VAR conditional mean. L_r is number of lags in ARCH conditional standard deviation. K_z is the degree of the square Hermite polynomial that captures the deviation of the standardized innovation z from conditional Gaussian distribution. The interaction polynomial term above the I_z degree is suppressed as zero. The degree of x -polynomial K_x is fixed at 0, and by convention L_p is set to be 1. l_θ is the number of coefficients in the SNP model. $s_n(\hat{\theta})$ is the negative sample mean log-likelihood. BIC is the Bayesian Information Criterion. According to BIC, the preferred SNP specification is 10514200.

1 In the standard 1-factor CIR model, the parameter estimates imply a highly
 2 persistent short-term interest rate process with a long-run average level of 6.4%
 3 [$\bar{r} = a_0/(-a_1)$] and a speed of adjustment of only 0.0907 ($\kappa = -a_1$). This roughly
 4 corresponds to an AR(1) process with the coefficient on the lagged interest rate
 5 being around 0.91, which is consistent with the results from the previous empirical
 6 studies of the interest rate. The estimates also imply stochastic volatility is an
 7 important property of the interest rate process with a conditional standard deviation
 of 1.27% on average ($\sqrt{\sigma\bar{r}}$).

Table 4. Parameter estimates of SNP density.

Parameter	Estimates	Standard Error
a(0,0)	1.00000	0.00000
a(0,1)	-0.05179	0.06782
a(1,0)	0.07111	0.06689
a(0,2)	-0.10675	0.05547
a(1,1)	0.22368	0.06511
a(2,0)	-0.23111	0.08141
a(0,3)	0.01787	0.01269
a(3,0)	-0.01448	0.01384
a(0,4)	0.01470	0.00595
a(4,0)	0.02222	0.00684
$\mu(2,0)$	-0.00079	0.00704
$\mu(1,0)$	-0.00531	0.01195
$\mu(2,2)$	0.98033	0.01447
$\mu(2,1)$	0.00659	0.01593
$\mu(1,2)$	0.01049	0.01401
$\mu(1,1)$	0.97699	0.01507
R(1,0)	0.04068	0.00901
R(2,0)	0.04830	0.00820
R(3)	0.08870	0.01251
R(1,1)	0.33931	0.07707
R(2,1)	0.18582	0.05988
R(1,2)	0.22569	0.07642
R(2,2)	0.11341	0.05650
R(1,3)	0.16133	0.06670
R(2,3)	0.25376	0.07700
R(1,4)	0.12867	0.05434
R(2,4)	0.03708	0.05448
R(1,5)	0.09542	0.05642
R(2,5)	0.00238	0.06357

This table reports point estimates as well as their standard errors of the parameters in the SNP model (10514200). $a(i, j)$ are parameters of the Hermit polynomial function. $\mu(i, j)$ are parameters of the VAR conditional mean. $R(i, j)$ are parameters of the ARCH standard deviation of the innovation z . See Gallant and Tauchen [26] or Bansal and Zhou [3] for more detailed interpretations of these parameters.

1 Model 2 introduces Markov regime shifts into the standard CIR model without
2 pricing such risk. Consistent with the previous studies, we find that the interest
3 rate process is characterized by two distinct regimes. In one regime (Regime 2) the
4 short rate $r(t)$ is highly persistent ($\kappa = -a_1 = 0.0735$) with a long-run mean of
5 11.7% [$\bar{r} = a_0/(-a_1)$]. The short rate in the other regime (Regime 1), however,
6 is less persistent ($\kappa = -a_1 = 0.1501$) with a much lower long-run mean of 1.8%.
7 Given the average short-term interest rate of 5% – 6% in the sample (see Table 1),
8 this implies that the interest rate is usually rising in Regime 2 and declining in
9 Regime 1. This empirical regularity is consistent with features of the business cycle

Table 5. Parameter estimates of the term structure models.

	Model 1	Model 2	Model 3
Regime 1			
a_0	0.0058 (0.0007)	0.0027 (0.0002)	0.0036 (0.0003)
a_1	-0.0907 (0.0096)	-0.1501 (0.0036)	-0.1488 (0.0006)
σ	0.0025 (0.0002)	0.0024 (2.6e-5)	0.0025 (0.0003)
θ_x	-10.7774 (1.7709)	-15.7689 (0.0070)	-15.5444 (0.0053)
θ_s			0.1438 (0.0337)
Regime 2			
a_0		0.0086 (0.0004)	0.0102 (0.0004)
a_1		-0.0735 (0.0022)	-0.0916 (0.0005)
σ		0.0030 (1.4e-5)	0.0034 (1.4e-5)
θ_x		-17.6468 (0.4841)	-16.9962 (0.0975)
θ_s			-0.0789 (0.0109)
Transition Intensity			
$\eta_s(1, 2)$		-1.2040 (0.0097)	-1.1655 (0.0263)
$\eta_s(2, 1)$		-1.4706 (0.0505)	-1.4457 (0.0130)
χ^2	213.34	109.70	60.38
Z-value	27.33	15.28	7.85
d.o.f.	24	18	16

This table reports EMM estimates of the term structure models. Model 1 refers to the standard 1-factor CIR without regime shifts. Model 2 refers to the 1-factor CIR model with regime shifts, but the risk of regime shifts is not priced. Model 3 is the 1-factor CIR model developed in Proposition 3.2 where the risk of regime shifts is priced. a_0 , a_1 and σ_0 are the coefficients in the diffusion process of $x(t)$: $dx = (a_0 + a_1 x)dt + \sqrt{\sigma x}dw$. θ_x is coefficient on the market price of diffusion risk, which is given by $\theta_x \sqrt{\sigma x}$. θ_s is the coefficient that determines the market price of regime-switching risk $\lambda_s(z)$, which is parameterized as $1 - e^{\theta_s(z)}$ in the model. The transition intensity of the Markov chain is given by $e^{\eta_s(z)}$. Numbers in parentheses are the standard errors. The table also reports the χ^2 statistics and its degree of freedom (dof) from the EMM estimation.

- 1 where expansions are usually characterized by rising interest rates and recessions
 2 tend to witness declining interest rates. Moreover, the estimated coefficients on the
 3 conditional volatility and the market price of (diffusion) risk all are different across
 regimes, further suggesting that the yield curve exhibits strong regime-dependence

1 properties.¹⁴ The regime-shift intensity is parameterized as $e^{\eta_s(z)}$ in the model.
 2 Table 5 reports that $\eta_s(1, 2) = -1.2040$ and $\eta_s(2, 1) = -1.4706$. This implies that
 3 Regime 2 is more persistent than Regime 1 (namely smaller transition probability
 4 from Regime 2 to Regime 1). For the monthly data used in the paper, the estimates
 5 of $\eta_s(z)$ suggest that the probability of switching from Regime 1 to Regime 2 is
 6 approximately 0.0250 while the probability of switching from Regime 2 to Regime 1
 7 is approximately 0.0191. These findings are consistent with the results based on
 8 discrete time models (e.g., [3]).

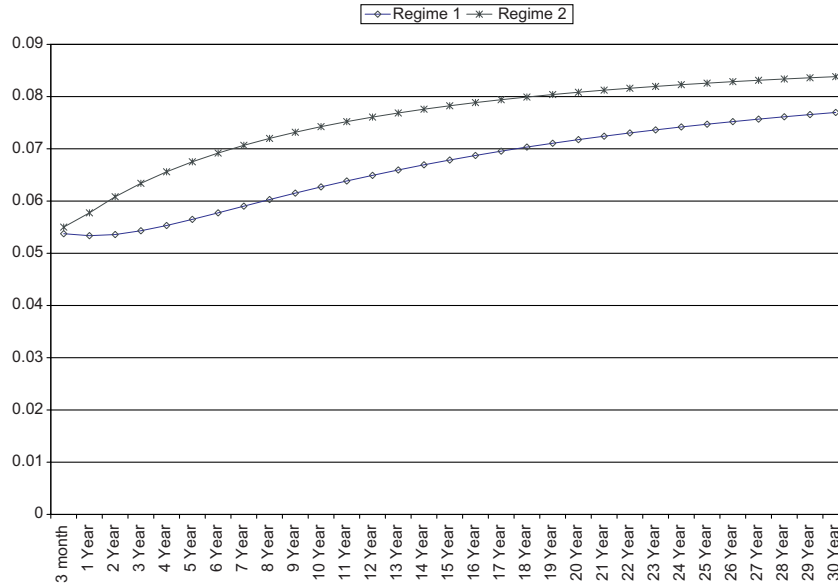
9 In Model 3, the risk of regime shifts is priced. The estimates of the model
 10 parameters ($a_0, a_1, \sigma, \theta_x$ and η_s) are similar to those obtained in Model 2, confirming
 11 that the periodic shifts across distinct regimes is an important empirical property of
 12 the interest rate dynamics. In Model 3, the implied long-run mean of $r(t)$ becomes
 13 11.1% in Regime 2 and 2.4% in Regime 1 respectively. The estimates of a_1 and σ
 14 indicate that $r(t)$ is more persistent with a larger conditional variance in Regime 2
 15 ($\kappa = -a_1 = 0.0916, \sigma = 0.0034$) than in Regime 1 ($\kappa = -a_1 = 0.1491, \sigma =$
 16 0.0025). As in Model 2, we also find that the market price of the diffusion risk varies
 17 across regimes. The estimate of θ_x is -15.54 in Regime 1 and -17.00 in Regime 2
 18 respectively. Finally, the estimates of $\eta_s(z)$ imply that, at monthly frequency, the
 19 Markov chain $s(t)$ switches from Regime 1 to Regime 2 with probability 0.0260 and
 20 switches from Regime 2 to Regime 1 with probability 0.0169.

21 In Fig. 1, we plot the yield curve at the estimated parameters in Regime 1
 22 together with that in Regime 2 by fixing the short-term interest rate at the sample
 23 average of 5.6% using Model 3. The differences in the yield curves are obvious. In
 24 Regime 2 the yield curve is higher and steeper compared to that in Regime 1 due
 25 to the fact that interest rates tend to rise in Regime 2. The average yield curve in
 26 Regime 1 not only has a lower level, but also has a different shape. It initially slopes
 27 downward and then slopes upward. This is because that the short-term interest rate
 28 declines in Regime 1 on average. Since the term premiums are small for bonds of
 29 short maturities, the interest rates on these bonds are mainly determined by the
 30 expectation of the short-term interest rate in the near future, therefore resulting in
 31 a negative slope in the yield curve. However, as maturities of the bonds increase,
 32 the term premiums start to play a more important role in determining the interest
 33 rates. Moreover, since Regime 1 is not as persistent as Regime 2, the short-term
 34 interest rate is also expected to increase over a long horizon as $s(t)$ switches from
 35 Regime 1 to Regime 2 in the future. Hence the slope of the yield curve becomes
 36 positive as maturities increase.

37 Figure 1 clearly shows that the yield curve alternates between two distinct
 38 regimes as $s(t)$ evolves over time. Whether or not these regime shifts pose a sig-
 39 nificant risk to investors depends on the estimate of $\theta_s(z)$, the parameter that
 40 determines the market price of the regime-switching risk.¹⁵ Table 5 shows that the

¹⁴The conditional volatility of the short term interest rate $r(t)$ is given by $\sqrt{\sigma r(t)}$, and the market price of diffusion risk is given by $\theta_x \sqrt{\sigma r(t)}$ in the model.

¹⁵The market price of the regime switching risk $\lambda_s(z)$ is parameterized as $1 - e^{\theta_s(z)}$ in the model, see Eq. (3.8).

18 *S. Wu & Y. Zeng*

The figure plots the estimated average yield curves (Model 3) in Regime 1 ($s(t) = 1$) and Regime 2 ($s(t) = 2$). When computing the yield curves, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%.

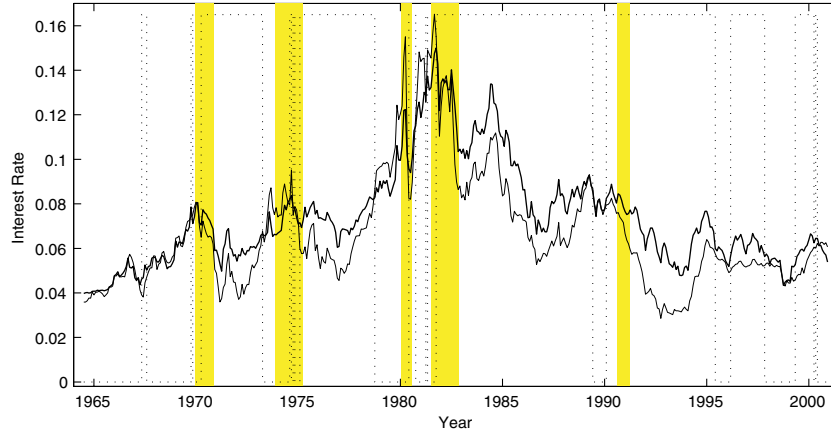
Fig. 1. Different regimes of the yield curves.

1 estimates of $\theta_s(z)$ are highly significant in both regimes. In particular, $\theta_s(z)$ is esti-
 2 mated to be 0.1438 with a standard error of 0.0337 in Regime 1 and -0.0789 with
 3 a standard error of 0.0109 in Regime 2 respectively. Moreover there is a substantial
 4 improvement in the goodness-of-fit of the regime-switching model. The χ^2 statistics
 5 decreases from 109.70 with 18 degree of freedom in Model 2 to 60.38 with 16 degree
 6 of freedom after the regime-switching risk is priced in Model 3.

7 Notice that the χ^2 statistics still indicates an overall poor fitting of the model.
 8 This is not surprising given that the model has only one factor (plus a Markov
 9 regime-switching variable). Bansal and Zhou [3] and Dai *et al.* [13] showed that the
 10 overall fit can be further improved in multi-factor models. Moreover, the results
 11 in Dai *et al.* [13] indicate that the market price of regime-switching risk remains
 12 significant even when additional factors are included. These results suggest that the
 13 significant regime-switching risk parameter found in our model is not due to omitted
 14 factors. The regime shifts in the term structure documented in many empirical
 15 studies most likely represent a systematic risk.

In fact, following the approach in Bansal and Zhou [3],¹⁶ we can plot the implied regimes together with the business cycle expansions and recessions identified by

¹⁶Specifically, the estimated term structure model allows us to compute interest rates of different maturities conditional on the regime $\hat{R}(t, \tau | s_t)$. An estimate of s_t can be obtained by choosing the regime that minimizes the average difference between the actually observed interest rate $R(t, \tau)$ and $\hat{R}(t, \tau | s_t)$, i.e., $\hat{s}_t = \arg \min_{\tau} |R(t, \tau) - \hat{R}(t, \tau | s_t)|$.



The figure plots the implied regimes by the term structure model (Model 3 in Table 5) together with the business cycle. NBER business cycle recessions are indicated the shaded area. The dashed line indicates the regimes implied by the interest rates. We also plot the 6-month (the thinner solid line) and 5-year interest rate (the thicker solid line) in the graph.

Fig. 2. Business cycle and the interest rate regimes.

1 NBER in Fig. 2. The figure clearly shows that the distinct regimes underlying
 2 the dynamics of the term structure of interest rates are intimately related to the
 3 fluctuations of the aggregate economy. Therefore it is important that these periodic
 4 shifts in regimes are treated as a systematic risk to investors in the dynamic term
 5 structure models. It is also interesting to note from Fig. 2 that the regime shifts
 6 tend to precede business cycle turning points.¹⁷ This is consistent with the empirical
 7 finding that the yield curve has significant predictive power for the business cycle,
 see Estrella and Mishkin [19] and Chauvet and Potter [8] among others.

9 4.3.1. The term premiums due to regime-switching risk

To assess quantitatively the potential impact of the regime-switching risk, we can decompose the term premium on a long-term bond into two parts. One is a term premium due to the diffusion risk and the other component is associated with the risk of regime shifts. In particular, ignoring the Jensen's inequality terms, one can obtain that,

$$\begin{aligned}
 R(t_0, \tau) - \frac{1}{\tau} E_{t_0} \left[\int_{t_0}^{t_0+\tau} r_t dt \right] &\approx \frac{1}{\tau} E_{t_0} \left[\int_{t_0}^{t_0+\tau} \theta_x(s_t) \sigma(s_t) x_t B(t_0 + \tau - t, s_t) dt \right] \\
 &+ \frac{1}{\tau} E_{t_0} \left[\int_{t_0}^{t_0+\tau} \int_E \lambda_s(z) (e^{\Delta_S A(t_0+\tau-t, s_t) + \Delta_S B(t_0+\tau-t, s_t) x_t} - 1) \gamma_m(dt, dz) \right],
 \end{aligned}
 \tag{4.1}$$

¹⁷Note that the regime shifts near the end of the sample period precede the most recent recession starting from March 2001 as classified by NBER.

20 *S. Wu & Y. Zeng*

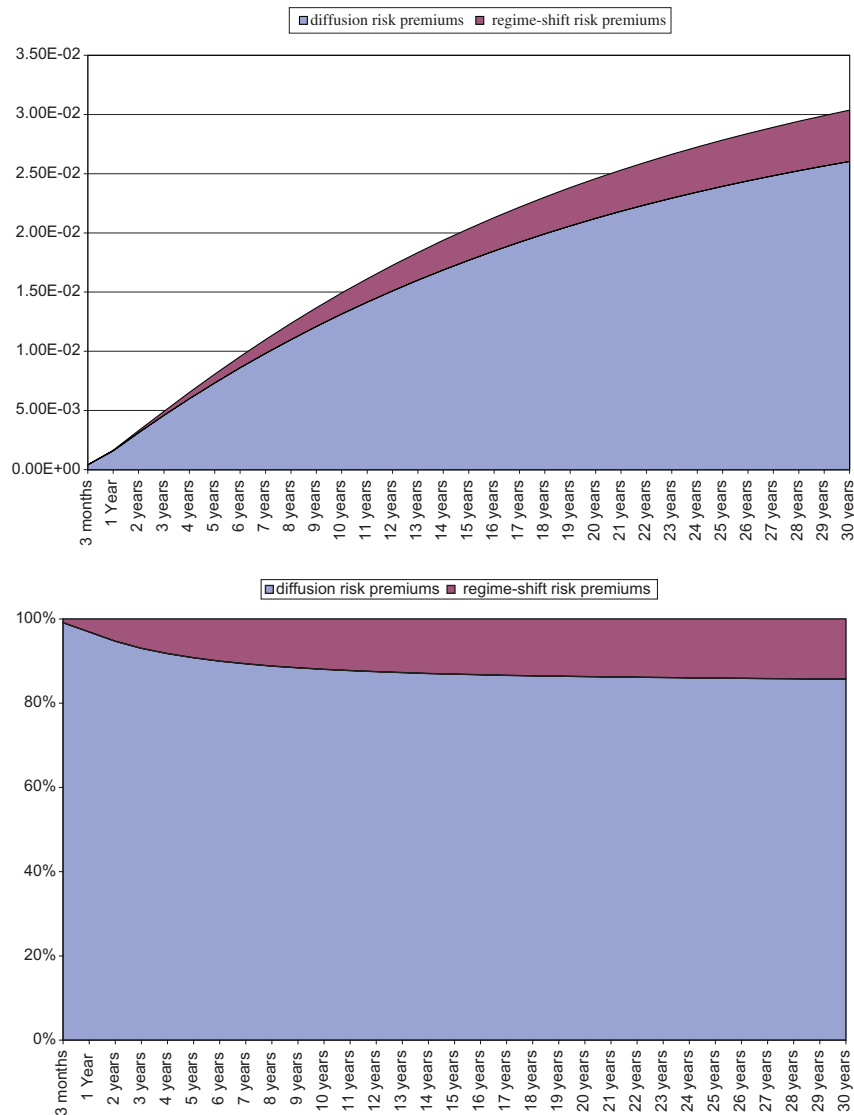
1 where $R(t_0, \tau)$ is the interest rate on a default-free bond of maturity τ at time
 2 t_0 , r_t is the instantaneous short-term interest rate. So the left-hand side of (4.1)
 3 is the difference between the interest rate on the long-term bond of maturity τ
 4 and the average of the expected future short-term interest rate. The Expectation
 5 Hypothesis of the term structure maintains that this difference is zero because long-
 6 term interest rates are solely determined by the expected future short rate. However,
 7 as it is made clear in (4.1), long-term interest rates can deviate substantially from
 8 the levels implied by the Expectation Hypothesis. The first term on the right-hand
 9 of (4.1) is the excess return demanded by investors due to the diffusion risk and
 10 the second term is the excess return due to the regime-switching risk, where the
 11 market prices of risk are given by $\theta_x(s_t)\sqrt{\sigma(s_t)x_t}$ and $\lambda_s(z)$ respectively. Note that
 12 $A(\tau, s)$ and $B(\tau, s)$ are determined in Proposition 3.2 and $\gamma_m(dt, dz)$ is given in
 13 (2.2). Using the estimated parameters, the average values of these two components
 14 of the term premium can be obtained through Monte Carlo simulations.¹⁸ In Fig. 3,
 15 we report the average total term premiums and the two components for bonds of
 16 various maturities ranging from three months to 30 years. In each graph the lower
 17 part measures the diffusion risk premiums and the upper part measures the regime-
 18 switching risk premiums. For bonds with short maturities (less than 1-year), we
 19 can see that the term premiums are small and are due to the diffusion risk. As
 20 bond maturity increases, both components of the term premiums increase, and
 21 the regime-switching risk premiums become a significant part of the total term
 22 premiums. The lower panel of Fig. 3 indicates that the regime-switching risk can
 23 account for more than 10% of the term premiums for bonds with maturities longer
 24 than six years, and up to 15% for a 30-year bond.

25 4.3.2. *How does the regime-shift risk affect the yield curve?*

26 The term structure model developed in Proposition 3.2 allows us to compute the
 27 prices of bonds of different maturities in the presence of the regime-switching risk.
 28 In this section we examine the impact of such risk on the yield curve. Specifi-
 29 cally, we compare the estimated bond prices in Model 3 with the bond prices
 30 obtained using the same model and the same parameter values except holding
 31 the market price of the regime-switching risk $\theta_s(z)$ at zero.¹⁹ When computing
 32 the bond prices, we set the short-term interest rate at the sample average of the
 33 1-month rate of 5.6%. We report the price differential between the two regimes in
 34 Fig. 4 and the corresponding yield curves in Fig. 5. Consistent with the result from
 35 the last section, we find that when fixing $\theta_s(z)$ at zero, the prices of long-term bonds

¹⁸More specifically, we simulate the 5,000 independent sample paths of $r(t)$ and $s(t)$ given $r(t_0)$ and $s(t_0)$. To get ride of the impact of the initial values of $r(t_0)$ and $s(t_0)$, for each sample path the first one thousand points of $r(t)$ and $s(t)$ are ignored. We then take the average over the 5,000 sample paths of $r(t)$ and $s(t)$ to obtain unconditional expectations.

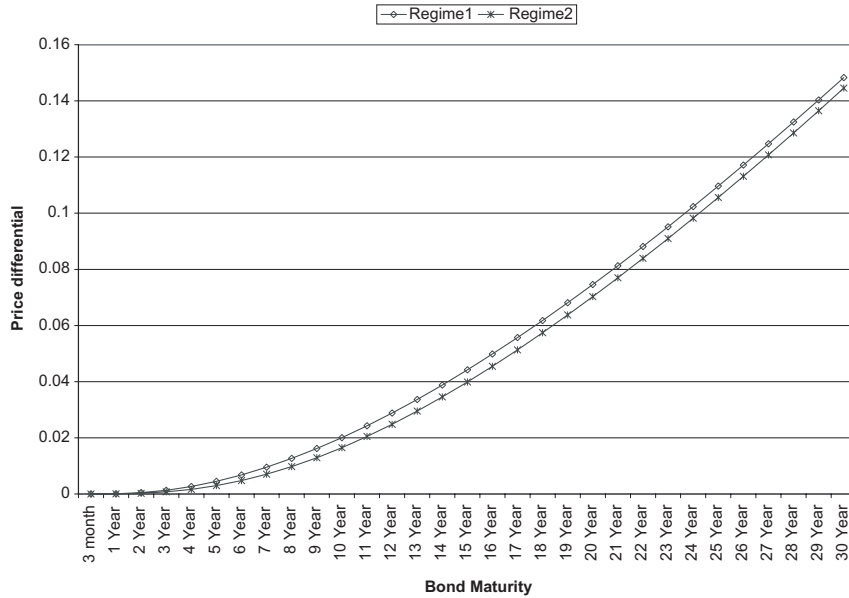
¹⁹In Model 3, the estimates of $\theta_s(z)$ are 0.1438 with a standard error of 0.0337 in Regime 1 and -0.0789 with a standard error of 0.0109 in Regime 2 respectively.



The above two figures report the decompositions of the term premiums for bonds of various maturities (from 3-month to 30-year). The upper panel plots the bond term premiums as the sum of two components due to the diffusion risk and the regime-shift risk respectively. The lower panel plots these two risk components as percentages of the total term premiums.

Fig. 3. Decompositions of the term premiums.

- 1 become significantly higher because of the reduction of the (total) term premiums.
- Figure 4 shows that, for example, the price of a 30-year bond would be about 15%
- 3 higher if $\theta_s(z)$ is set to zero instead of its estimated value of 0.1438 in Regime 1 or -0.0789 in Regime 2. On the other hand, the regime-switching risk has almost no

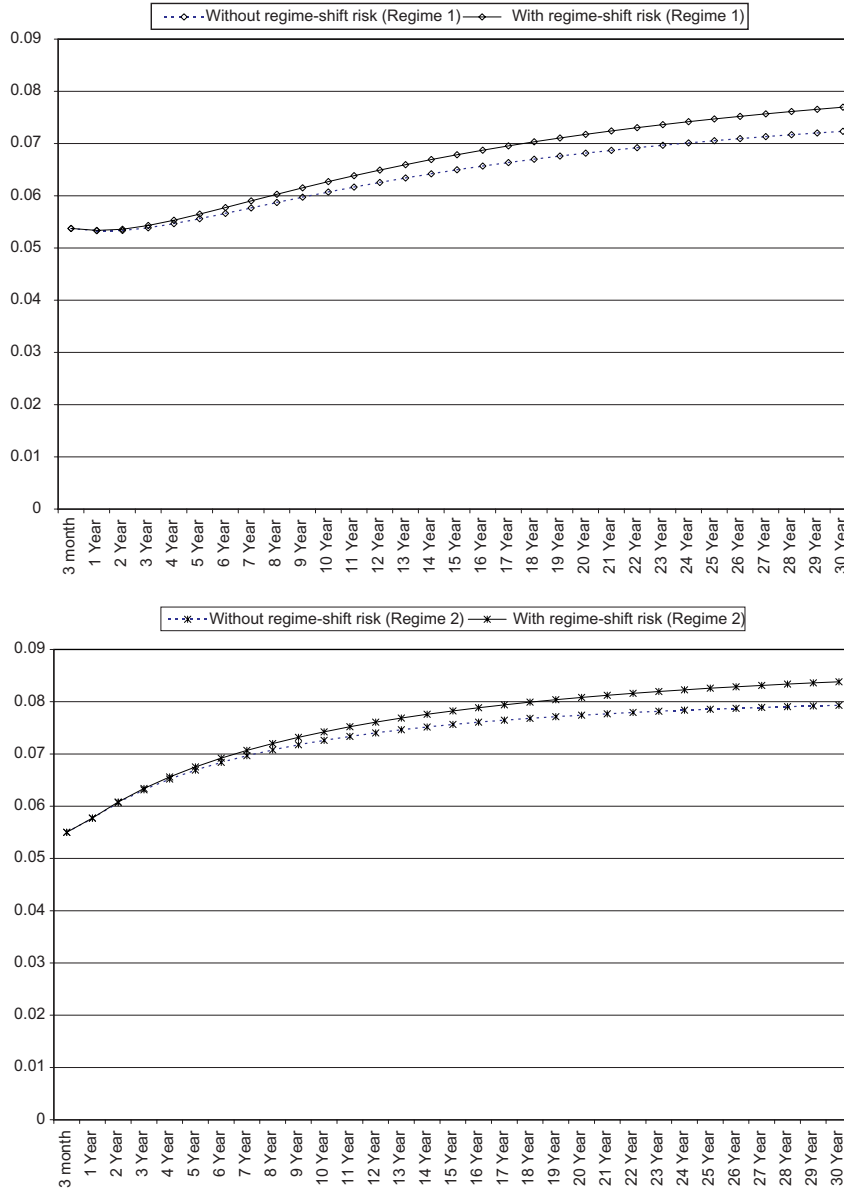


The figure reports the impact of regime-shift risk on bond prices. The price differential is obtained as $\frac{P^*(\tau) - P(\tau)}{P(\tau)}$, where $P(\tau)$ is the estimated price of the bond of maturity τ in Model 3. $P^*(\tau)$ is the price of the same bond obtained using the same model and the same parameter values except holding the market price of regime-shift risk at zero ($\theta_s(z) = 0$). In calculating the bond prices, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%. The line with diamonds is the bond price differential in Regime 1. The line with crosses is the bond price differential in Regime 2.

Fig. 4. The impact of regime-shift risk on bond prices.

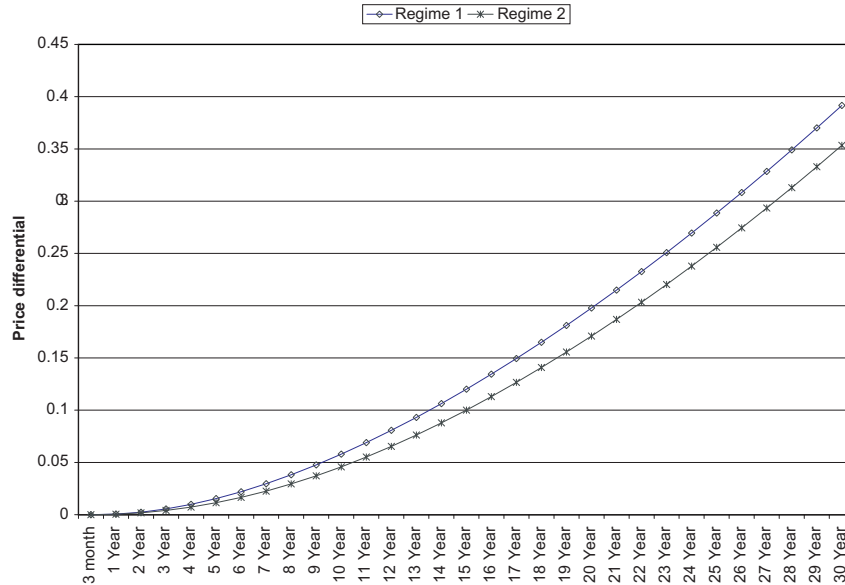
1 effect on short-term bonds. These result implies that the regime-shift risk mostly
 2 affects the long end of the yield curve as expected. Ignoring the regime-shift risk
 3 would lead to underestimation of long-term interest rates, and therefore flatter yield
 4 curves (see Fig. 5).

5 Another way to examine the impact of the regime-shift risk on the yield curve is
 6 to compare the bond prices estimated by Model 2 and Model 3 respectively. The two
 7 models are the same 1-factor CIR model subject to Markov regime changes except
 8 that Model 2 does not price the risk of such regime shifts. To compute the bond
 9 prices, we again fix the short-term interest rate $r(t)$ at the sample average of the
 10 1-month rate of 5.6%. We report in Fig. 6 the bond price differential in both regimes.
 11 The corresponding yield curves are plotted in Fig. 7. The results are similar to those
 12 obtained above. In particular, for bonds of short maturities (less than three years),
 13 the two term structure models give almost the same result, suggesting that the risk
 14 of regime shifts is not important for short-term bonds. However, as bond maturities
 15 increase, the price differential becomes significantly bigger. In fact, Model 2 implies
 16 much higher prices (up to 40% higher in Regime 1 and 35% higher in Regime 2)
 17 than Model 3 does for bonds of long maturities. The corresponding difference in the



These two figures illustrate the impact of regime-shift risk on the yield curve in Regime 1 (the upper panel) and Regime 2 (the lower panel) respectively. The solid line is the estimated yield curve from Model 3. The dashed line is the yield curve obtained using the same model and the same parameter values except holding the market price of regime-shift risk at zero ($\theta_s(z) = 0$). In calculating the yields, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%.

Fig. 5. The impact of regime-shift risk on the yield curve.



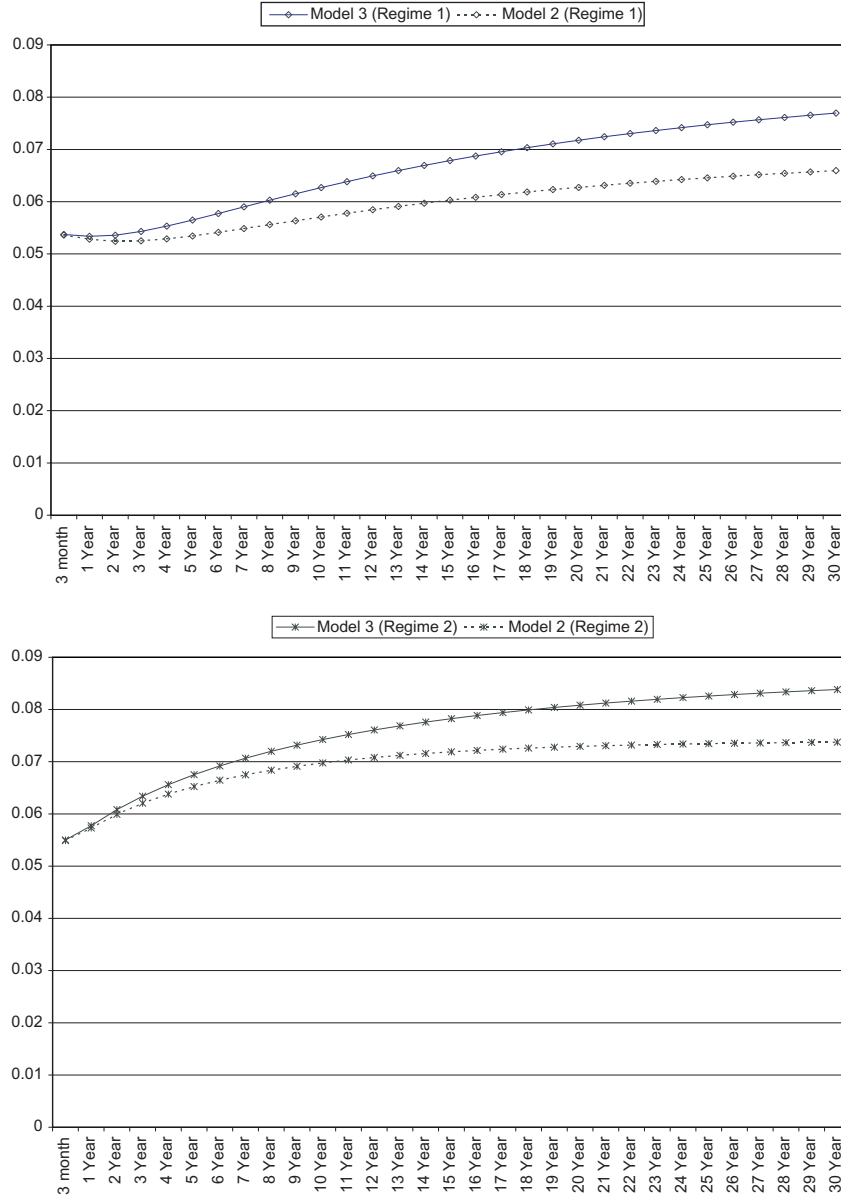
The price differential is obtained as $\frac{P(\text{Model 2}) - P(\text{Model 3})}{P(\text{Model 3})}$, where $P(\text{Model 3})$ is the estimated bond price using Model 3. $P(\text{Model 2})$ is the estimated price of the same bond using Model 2. The difference between the two models is that in Model 2 the risk of regime shifts is not priced. In calculating the bond prices, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%. The line with diamonds is the bond price differential in Regime 1. The line with crosses is the bond price differential in Regime 2.

Fig. 6. Bond price differentials.

1 yield curves can be clearly seen in Fig. 7. Without pricing the risk of regime shifts,
 2 Model 2 obtains much flatter yield curves in both regimes than Model 3 does.

3 **5. Concluding Remarks**

4 Previous studies have provided strong empirical evidence that the joint movement
 5 of interest rates of different maturities can be well described by dynamic term struc-
 6 ture models that incorporate regime shifts. Moreover these studies also show that
 7 there is a close link between the regime shifts and the business cycle fluctuations.
 8 Therefore it is very likely that such large periodic shifts of interest rates across
 9 distinct regimes present a systematic risk to investors. This paper elucidates such
 10 regime-switching risk in a general equilibrium model of the term structure of inter-
 11 est rates. We show that the regime-switching risk premium depends on the covari-
 12 ations between the discreet changes in marginal utility and bond prices. This new
 13 component therefore introduces an additional source of time-variation in bond risk
 14 premiums and can offer additional econometric flexibility to account for the joint
 15 movements of interest rates with different maturities. Our empirical results suggest
 16 that the regime-switching risk is a significant factor determining bond prices and
 17 has a major impact particular on the long-end of the yield curve.



These two figures report the average yield curves estimated by Model 2 and Model 3 respectively in Regime 1 (the upper panel) and in Regime 2 (the lower panel). The difference between the two models is that Model 2 does not price the risk of regime shifts. The solid line is the estimated yield curve from Model 3. The dashed line is the estimated yield curve from Model 2. When calculating the yields, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%.

Fig. 7. Differences in the yield curves.

26 *S. Wu & Y. Zeng*

1 One caveat of the current paper is that we have to rely on log-linear approx-
 2 imations to obtain a closed-form solution to the term structure of interest rates.
 3 How important the approximation errors are is a question we are currently inves-
 4 tigating. Nevertheless, we notice that relaxing some of the assumptions (such as
 5 regime-dependent factor loadings) can lead to exact solutions of the yield curve.
 6 Another caveat to note is that the estimated 1-factor term structure model has a
 7 poor empirical fitting. We do not pretend that the term structure is fully charac-
 8 terized by this single factor. We use that model mainly to illustrate the potential
 9 impact of a systematic regime-switching risk on the yield curve.

10 This systematic risk of regime shifts is also likely to have important implications
 11 for pricing bond derivatives (e.g., [38]) as well as for investors' optimal portfolio
 12 choice problem (e.g., [7]). These are left for future research. Another extension is
 13 to estimate the term structure model jointly with macroeconomic variables under
 14 regime shifts. This would provide more direct evidence regarding the nature of the
 15 regime-switching risk.

Appendix A: Proof of Proposition 2.1

17 Let $J(w(t), s(t), x(t)) = \sup E_t \left[\int_t^\infty e^{-\rho(\tau-t)} U(c(\tau)) d\tau \right]$. We assume that a solu-
 18 tion to the agent's problem exists and indirect utility function $J(w(t), s(t), x(t))$
 19 as well as the optimal consumption and portfolio choice satisfy the Bellman equa-
 20 tion (see the result in Sec. 3.1.6 of [31])

$$21 \quad \sup_{(\phi_1, \phi_2, c)} \mathcal{D}J(w, s, x) - \rho J(w, s, x) + U(c) = 0, \quad (\text{A.1})$$

where

$$\begin{aligned} \mathcal{D}J(w, s, x) &= (w\mu_w)J_w + \mu_x J_x + \frac{1}{2}(w\sigma_w)^2 J_{ww} + (w\sigma_w)\sigma_x J_{wx} \\ &\quad + \frac{1}{2}\sigma_x^2 J_{xx} + \int_E \Delta_s J \gamma_m(dz), \end{aligned}$$

and

$$\Delta_s J = J(w(1 + \delta_w(z)), s + \zeta(z), x) - J(w, s, x).$$

23 In the above equations, $J_w = \frac{\partial J}{\partial w}$, $J_x = \frac{\partial J}{\partial x}$, $J_{ww} = \frac{\partial^2 J}{\partial w^2}$, $J_{wx} = \frac{\partial^2 J}{\partial w \partial x}$. μ_w , σ_w
 24 and δ_w are given in (2.12), (2.13) and (2.14) respectively. $\gamma_m(dz)$ is defined in (2.17).
 25 μ_x and σ_x are the drift and diffusion terms of the state variable $x(t)$ respectively
 as defined in (2.1).

The first order conditions for (A.1) are

$$27 \quad U'(c) - J_w = 0, \quad (\text{A.2})$$

$$\begin{aligned} &w(\mu_y - r)J_w + (w\sigma_w)(w\sigma_x)J_{wx} + (w\sigma_y)\sigma_w J_{wy} \\ &\quad + \int_E w\delta_y(z)J_w(w(1 + \delta_w(z)), s + \zeta(z), x)\gamma_m(dz) = 0, \quad (\text{A.3}) \end{aligned}$$

$$w(\mu_F - r)J_w + (w\sigma_w)(w\sigma_F)J_{ww} + (w\sigma_F)\sigma_x J_{wx} + \int_E w\delta_F(z)J_w(w(1 + \delta_w(z)), s + \zeta(z), x)\gamma_m(dz) = 0. \quad (\text{A.4})$$

1 Note that at equilibrium $\phi_1 = 1$ and $\phi_2 = 0$, hence Proposition 2.1 follows from (A.3).

3 Appendix B: Proof of Proposition 2.2

From (A.4) above, we have

$$\begin{aligned} \mu_F + \int_E \delta_F(z)\gamma_m(dz) - r &= \left(-\frac{J_{ww}}{J_w}\right) \frac{(w\sigma_w)(w\sigma_F)}{w} + \left(-\frac{J_{wx}}{J_w}\right) \frac{(w\sigma_F)\sigma_x}{w} \\ &+ \int_E \left(-\frac{\Delta_s J_w}{J_w}\right) \frac{\Delta_s F}{F} \gamma_m(dz). \end{aligned} \quad (\text{B.1})$$

Applying Ito's formula to $F(t, w, s, x)$, we have²⁰

$$\begin{aligned} dF &= \left[F_t + (w\mu_w)F_w + \mu_x F_x + \frac{1}{2}(w\sigma_w)^2 F_{ww} + (w\sigma_w)\sigma_x F_{wx} + \frac{1}{2}\sigma_x^2 F_{xx} \right] dt \\ &+ [(w\sigma_w)F_w + \sigma_x F_x] dB(t) + \int_E \Delta_s F m(dt, dz). \end{aligned} \quad (\text{B.2})$$

Comparing (B.2) with (2.7), we have

$$5 \quad F\mu_F = F_t + (w\mu_w)F_w + \mu_x F_x + \frac{1}{2}(w\sigma_w)^2 F_{ww} + (w\sigma_w)\sigma_x F_{wx} + \frac{1}{2}\sigma_x^2 F_{xx}, \quad (\text{B.3})$$

$$F\sigma_F = (w\sigma_w)F_w + \sigma_x F_x, \quad (\text{B.4})$$

7 and

$$\delta_F(z) = \frac{\Delta_s F}{F}. \quad (\text{B.5})$$

9 Proposition 2.1 follows after substituting (B.4) and (B.5) into (B.1) and defining $\mu_F^* = \mu_F + \int_E \delta_F(z)\gamma_m(dz)$.

11 Appendix C: Proof of Proposition 3.1

Proposition 2.2 implies that

$$\begin{aligned} \mu_F + \int_E \delta_F(z)\gamma_m(dz) - r &= \left[\left(-\frac{J_{ww}}{J_w}\right) \text{Var}(w) + \left(-\frac{J_{wx}}{J_w}\right) \text{Cov}(w, x) \right] \frac{F_w}{F} \\ &+ \left[\left(-\frac{J_{ww}}{J_w}\right) \text{Cov}(w, x) + \left(-\frac{J_{wx}}{J_w}\right) \text{Var}(x) \right] \frac{F_x}{F} \\ &+ \int_E \left(-\frac{\Delta_s J_w}{J_w}\right) \frac{\Delta_s F}{F} \gamma_m(dz). \end{aligned} \quad (\text{C.1})$$

²⁰See Protter [36] for the generalized Ito's formula for semi-martingales.

28 *S. Wu & Y. Zeng*

Using (B.3), we have

$$\begin{aligned}
F_t &+ \frac{1}{2}(w\sigma_w)^2 F_{ww} + (w\sigma_w)\sigma_x F_{wx} + \frac{1}{2}\sigma_x^2 F_{xx} \\
&+ \left[\mu_x - \left(-\frac{J_{ww}}{J_w} \right) (w\sigma_w)\sigma_x - \left(-\frac{J_{wx}}{J_w} \right) \sigma_x^2 \right] F_x \\
&+ \left[(w\mu_w) - \left(-\frac{J_{ww}}{J_w} \right) (w\sigma_w)^2 - \left(-\frac{J_{wx}}{J_w} \right) (w\sigma_w)\sigma_x \right] F_w \\
&+ \int_E \left(1 + \frac{\Delta_s J_w}{J_w} \right) \Delta_s F \gamma_m(dz) = rF.
\end{aligned} \tag{C.2}$$

Note again that at equilibrium, $\phi_1 = 1$ and $\phi_2 = 0$. Hence (2.12), (2.13) and (2.14) imply that

$$w\mu_w = w\mu_y - c, \tag{C.3}$$

$$w\sigma_w = w\sigma_y, \tag{C.4}$$

$$w\delta_w(z) = w\delta_y(z). \tag{C.5}$$

Moreover, Proposition 2.1 implies

$$\begin{aligned}
wr &= w\mu_y - \left(-\frac{J_{ww}}{J_w} \right) (w\sigma_y)^2 - \left(-\frac{J_{wx}}{J_x} \right) (w\sigma_y)\sigma_x \\
&+ \int_E \left(1 + \frac{\Delta_s J_w}{J_w} \right) w\delta_y(z)\gamma_m(dz).
\end{aligned} \tag{C.6}$$

Combining (C.2)–(C.6), We have the following fundamental partial differential equations of asset pricing as in CIR

$$\begin{aligned}
F_t &+ \frac{1}{2}(w\sigma_y)^2 F_{ww} + (w\sigma_y)\sigma_x F_{wx} + \frac{1}{2}\sigma_x^2 F_{xx} \\
&+ \left[\mu_x - \left(-\frac{J_{ww}}{J_w} \right) (w\sigma_y)\sigma_x - \left(-\frac{J_{wx}}{J_w} \right) \sigma_x^2 \right] F_x \\
&+ \left[wr - c - \int_E \left(1 + \frac{\Delta_s J_w}{J_w} \right) w\delta_y(z)\gamma_m(dz) \right] F_w \\
&+ \int_E \left(1 + \frac{\Delta_s J_w}{J_w} \right) \Delta_s F \gamma_m(dz) = rF.
\end{aligned} \tag{C.7}$$

- 1 Under logarithm utility function $U(c(t)) = \log c(t)$, it is well known that the
indirect utility function is separable in $w(t)$ and $x(t)$ and $s(t)$, i.e., $J(w, s, x)$ can be
3 written as $\frac{1}{\rho} \log w + f(s, x)$ where $f(s, x)$ solves the system of differential equation
after substituting $J(w, s, x)$ and the optimal choice of consumption (c^*) and port-
5 folio (ϕ_1^*, ϕ_2^*) into the Bellman Eq. (A.1). This separability implies that $J_{wx} = 0$.

1 Moreover, for default-free discount bonds, $F_w = 0$, $F_{ww} = 0$ and $F_{wx} = 0$.
 Therefore Eq. (C.7) can be simplified as

$$3 \quad F_t + \frac{1}{2}\sigma_x^2 F_{xx} + (\mu_x - \sigma_y \sigma_x) F_x + \int_E \left(1 + \frac{\Delta_s J_w}{J_w}\right) \Delta_s F \gamma_m(dz) = rF. \quad (\text{C.8})$$

Using the fact that $J_w = \frac{1}{\rho_w}$ and (C.5), it can be easily shown

$$5 \quad 1 + \frac{\Delta_s J_w}{J_w} = 1 - \lambda_s(z), \quad (\text{C.9})$$

where $\lambda_s(z) = \frac{\delta_y(z)}{1+\delta_y(z)}$.

7 Proposition 3.1 can be obtained by substituting the above equation into (C.8).
 Note that (C.8) defines a system of partial differential equations.

9 Appendix D: Proof of Proposition 3.2

Without loss of generality, let the price at time t of a pure-discount bond that will
 mature at T be given as

$$F(t, s(t), r(t), T) = e^{A(\tau, s(t)) + B(\tau, s(t))r(t)},$$

where $\tau = T - t$ and $A(0, s) = 0$, $B(0, s) = 0$.

Proposition 3.2 then implies

$$\begin{aligned} r = & -\frac{\partial A(\tau, s)}{\partial \tau} - \frac{\partial B(\tau, s)}{\partial \tau} r \\ & + [a_0(s) + (a_1(s) - \theta_x(s)\sigma(s))r]B(\tau, s) + \frac{1}{2}[\sigma(s)r]B^2(\tau, s) \\ & + \int_E (e^{\Delta_s A + \Delta_s B r} - 1) e^{\eta_s(z) + \theta_s(z)} \mathbf{1}(s = i) \epsilon_z(dz), \end{aligned} \quad (\text{D.1})$$

11 where $\Delta_s A = A(\tau, s + \zeta(z)) - A(\tau, s)$ and $\Delta_s B = B(\tau, s + \zeta(z)) - B(\tau, s)$.

Since r is small, applying log-linear approximation to $e^{\Delta_s B r}$, we have

$$e^{\Delta_s B r} \approx 1 + \Delta_s B r.$$

13 Proposition 3.2 follows by substituting the above equation into (D.1) and match
 the coefficients on r on both sides of the equation.

Acknowledgments

15 We are grateful to an anonymous referee as well as Philip Dybvig, Tom Kurtz,
 Ming Liu, Cyrus Ramezani, and Ken Singleton for comments and suggestions.
 17 We also thank participants at 2004 IMA Workshop of Financial Engineering on
 Model Implementation, Algorithms and Software Issues, the 11th Annual Global
 19 Finance Conference (2004), 2003 Missouri Economics Conference, 2003 AMS-IMS-
 SIAM Summer Research Conference on Mathematics of Finance, and at seminars
 21 of the University of Wisconsin at Madison and the University of Kansas for help-
 ful comments. Part of the work was done while the first author was visiting City

30 *S. Wu & Y. Zeng*

1 University of Hong Kong, whose hospitality is greatly appreciated. Ikuyasu Usui
provided able research assistance.

3 **References**

- 5 [1] C. M. Ahn and H. E. Thompson, Jump-diffusion processes and the term structure of
interest rates, *The Journal of Finance* **43** (1988) 155–174.
- 7 [2] A. Ang and G. Bekaert, Regime switches in interest rates, NBER working paper 6508
(1998).
- 9 [3] R. Bansal and H. Zhou, Term structure of interest rates with regime shifts, *The
Journal of Finance* **57** (2002) 1997–2043.
- 11 [4] R. Bansal, R. Gallant and G. Tauchen, Nonparametric estimation of structural mod-
els for high-frequency currency market data, *Journal of Econometrics* **66** (1995)
251–287.
- 13 [5] T. Bielecki and M. Rutkowski, Modeling of the defaultable term structure: Condi-
tional Markov Approach, working paper, The Northeastern Illinois University (2001).
- 15 [6] J. Boudoukh *et al.*, Regime shifts and bond returns, working paper, New York
University (1999).
- 17 [7] J. Y. Campbell and L. M. Viceira, Who should buy long-term bonds, *The American
Economic Review* **91** (2001) 99–127.
- 19 [8] M. Chauvet and S. Potter, Forecasting recessions using the yield curve, to appear in
Journal of Forecasting.
- 21 [9] J. Cox, J. Ingersoll and S. Ross, An intertemporal general equilibrium model of asset
prices, *Econometrica* **53** (1985) 363–384.
- 23 [10] J. Cox, J. Ingersoll and S. Ross, A theory of the term structure of interest rates,
Econometrica **53** (1985) 385–407.
- 25 [11] Q. Dai and K. Singleton, Specification analysis of affine term structure models, *Jour-
nal of Finance* **LV** (2000) 1943–1978.
- 27 [12] Q. Dai and K. Singleton, Term structure dynamics in theory and reality, *The Review
of Financial Studies* **16** (2003) 631–678.
- 29 [13] Q. Dai, K. J. Singleton and W. Yang, Regime shifts in a dynamic term structure
model of U.S. Treasury bond yields, working paper, New York University (2003).
- 31 [14] G. Di Masi, Yu. Kabanov and W. Runggaldier, Mean-variance hedging of options
on stocks with Markov volatilities, *Theory of Probability and Its Applications* **39**(1)
33 (1994) 172–182.
- 35 [15] F. Diebold and G. Rudebusch, Measuring business cycles: A modern perspective,
Review of Economics and Statistics **78** (1996) 67–77.
- 37 [16] D. Duffie and R. Kan, A yield-factor model of interest rates, *Mathematical Finance*
6 (1996) 379–406.
- 39 [17] D. Duffie, J. Pan and K. Singleton, Transform analysis and asset pricing for affine
jump-diffusions, *Econometrica* **68** (2000) 1343–1376.
- 41 [18] R. J. Elliott and R. S. Mamon, Term structure of a Vasicek model with a Markovian
mean reverting level, working paper, University of Calgary.
- 43 [19] A. Estrella and F. S. Mishkin, The term structure of interest rates and its role in
monetary policy for the European Central Bank, NBER working paper, No.5279
(1995).
- 45 [20] M. Evans, Real risk, inflation risk, and the term structure, working paper, Georgetown
University (2001).
- 47 [21] E. Fama and K. French, Business conditions and expected returns on bonds and
stocks, *Journal of Financial Economics* **25** (1989) 23–49.

- 1 [22] A. J. Filardo, Business cycle phases and their transitional dynamics, *Journal of Business and Economic Statistics* **12** (1994) 299–308.
- 3 [23] R. Garcia and P. Perron, An analysis of the real interest rate under regime shifts, *Review of Economics and Statistics* **78** (1996) 111–125.
- 5 [24] R. Gallant and G. Tauchen, Which moment to match? *Econometric Theory* **12** (1996) 657–681.
- 7 [25] R. Gallant and G. Tauchen, Reprojecting partially observed system with application to interest rate diffusions, *Journal of the American Statistical Association* **93** (1998) 10–24.
- 9 [26] R. Gallant and G. Tauchen, Efficient method of moments, working paper, University of North Carolina (2001).
- 11 [27] X. Guo, J. Miao and E. Morellec, Irreversible investment with regime shifts, to appear in *Journal of Economic Theory*.
- 13 [28] S. Gray, Modeling the conditional distribution of interest rates as a regime-switching process, *Journal of Financial Economics* **42** (1996) 27–62.
- 15 [29] J. Hamilton, Rational expectations econometric analysis of changes in regimes: An investigation of the term structure of interest rates, *Journal of Economic Dynamics and Control* **12** 385–423.
- 17 [30] J. Hamilton, A new approach to the economic analysis of nonstationary time series and the business cycle, *Econometrica* **57** (1989) 357–384.
- 19 [31] H. J. Kushner and P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd edition (Springer, New York, 2001).
- 21 [32] C. Landen, Bond pricing in a hidden Markov model of the short rate, *Finance and Stochastics* **4** (2000) 371–389.
- 23 [33] G. Last and A. Brandt, *Marked Point Processes on the Real Line* (Springer, New York, 1995).
- 25 [34] X. Mao, C. Yuan and G. Yin, The Euler–Maruyama method for stochastic differential equations with Markovian switching under non-Lipschitz conditions, working paper, University of Wales Swansea (2005).
- 27 [35] V. Naik and M. H. Lee, Yield curve dynamics with discrete shifts in economic regimes: Theory and estimation, working paper, University of British Columbia (1997).
- 29 [36] P. Protter, *Stochastic Intergration and Differential Equations* (Springer Verlag, Berlin, 1990).
- 31 [37] S. Scott, Bayesian methods for hidden Markov models: Recursive computing in the 21st century, *Journal of the American Statistical Association*, **97** (2002) 337–351.
- 33 [38] K. J. Singleton and L. Umantsev, Pricing coupon-bond options and swaptions in affine term structure models, *Mathematical Finance* **12** (2002) 427–446.
- 35 [39] A. Timmermann, Moments of Markov switching models, *Journal of Econometrics* **96** (2000) 75–111.
- 37
- 39