

# Affine Regime-Switching Models for Interest Rate Term Structure

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**ABSTRACT.** To model the impact of the business cycle, this paper develops a tractable dynamic term structure model under diffusion and regime shifts with time varying transition probabilities. The model offers flexible parameterization of the market prices of risk, including the price of regime switching risk. Closed form solutions for the term structure of interest rate are obtained for both affine- and quadratic-type models using log-linear approximation.

## 1. Introduction

There is strong empirical evidence suggesting that the aggregate economy is characterized by periodic shifts between distinct regimes of the business cycle (e.g. Hamilton [21], Filardo [17] and Diebold and Rudebusch [11]). A number of papers have also successfully used Markov regime-switching models to fit the dynamics of the short-term interest rate (see, among others, Hamilton [20], Garcia and Perron [18], Gray [19] and Ang and Bekaert [2]).<sup>1</sup> These results have motivated the recent studies of the impact of regime shifts on the entire yield curve using dynamic term structure models. A common approach, as in Naik and Lee [24], Boudoukh et al. [6], Evans [16] and Bansal and Zhou [3], is to incorporate Markov-switching into the processes of the pricing kernel and/or state variables. The regime-dependence introduced by these papers implies richer dynamic behavior of the market prices of risk and therefore offers greater econometric flexibility for the term structure models to simultaneously account for the time series and cross-sectional properties of interest rates. However, as pointed out by a recent survey paper by Dai and Singleton [10], the risk of regime shifts is not priced in these models, hence does not contribute independently to bond risk premiums.

The purpose of the present paper is to develop a tractable latent factor model that can capture the effects of regime-switching, especially, the systematic risk of regime switching. This paper draws from the recent literature on dynamic term

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<sup>1</sup>The expectation theory is usually invoked to relate long-term interest rates to the short rate in this literature, such as in Ang and Bekaert [2].

structure models under regime shifts. The main difference between the current paper and the previous studies is that the risk of regime shifts is explicitly priced in our model. The previous studies have all ignored the regime shift risk premiums except the recent paper by Dai and Singleton [10]. Here we show that closed-form solution can be obtained using log-linear approximation for both affine- and quadratic-type term structure models which also price the regime shifting risk. Our model implies that bond risk premiums include two components under regime shifts in general. One is a regime-dependent risk premium due to diffusion risk as in the previous studies. This risk premium has added econometric flexibilities relative to those in single-regime models because of the Markov-shift of the underlying parameters. The other is a regime-switching risk premium that depends on the difference of bond prices across regimes as well as the Markov transition probabilities. Therefore the model introduces a new source of time-variation in bond risk premiums. This additional component of the term premiums is associated with the systematic risk of periodic shifts in bond prices due to regime changes. Given the empirical evidence from the previous studies that the yield curve exhibits significantly different properties across regimes, the model implies that the regime-switching risk is likely to be an important factor that affects bond returns.

The remainder of this paper is organized as follows. We develop a tractable dynamic term structure models under diffusion and regime shifts in section 2. We derive closed-form solutions for the term structure using log-linear approximation for affine-type as well as quadratic-type term structure models with regime switching in section 3. Section 4 contains some concluding remarks.

## 2. The Model

Consider a two-factor model of the term structure of interest rates.<sup>2</sup> The two factors are the instantaneous short term interest rate  $r_t$  and the regime,  $s_t$ , where  $s_t$  is a  $(N + 1)$ -state continuous-time Markov chain taking on value of  $0, 1, 2, \dots, N$ .

**2.1. Modeling Regime Shifting Process.** In the literature of interest rate term structure, there are three approaches to model regime shifting process. The first approach is the *Hidden Markov Model*, summarized in the book of Elliott et. al. [14], and its application to the term structure can be found in Elliott and Mamon [15]. The second approach is the *Conditional Markov Chain*, discussed in Yin and Zhang [28], and its applications to the term structure are in Bielecki and Rutkowski ([4],[5]). The third approach is the *Marked Point Process* as in Landen [22]. Here, we adopt the marked point process approach due to its notational simplicity.

Following Landen [22], we define the mark space  $E$  as

$$E = \{(i, j) : i \in \{0, 1, \dots, N\}, j \in \{0, 1, 2, \dots, N\}, i \neq j\},$$

which includes all possible regime switchings and  $z = (i, j)$  is a generic point in  $E$ . We denote  $E$ 's  $\sigma$ -algebra as  $\mathcal{E} = 2^E$ . A marked point process,  $m(t, \cdot)$  is uniquely characterized by its stochastic intensity kernel,<sup>3</sup> which can be defined as

$$(2.1) \quad \gamma_m(dt, dz) = h(z, x(t-))\mathbf{I}\{s(t-) = i\}\epsilon_z(dz)dt,$$

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<sup>2</sup>We can easily generalize the model to include more factors. We only consider two factors for exposition purpose.

<sup>3</sup>See Last and Brandt [23] for detailed discussion of marked point process, stochastic intensity kernel and related results.

where  $h(z, x(t-))$  is the regime-shift (from regime  $i$  to  $j$ ) intensity at  $z = (i, j)$  (We assume  $h(z, x(t-))$  is bounded in  $[0, T]$ ),  $\mathbf{I}\{s(t-) = i\}$  is an indicator function, and  $\epsilon_z(A)$  is the Dirac measure (on a subset  $A$  of  $E$ ) at point  $z$  (defined by  $\epsilon_z(A) = 1$  if  $z \in A$  and 0, otherwise). Heuristically, for  $z = (i, j)$ ,  $\gamma_m(dt, dz)$  can be thought of as the conditional probability of shifting from Regime  $i$  to Regime  $j$  during  $[t, t+dt)$  given  $x(t-)$  and  $s(t-) = i$ .

Let  $A$  be a subset of  $E$ . Then  $m(t, A)$  counts the cumulative number of regime shifts that belong to  $A$  during  $(0, t]$ .  $m(t, A)$  has its compensator,  $\gamma_m(t, A)$ , given by

$$(2.2) \quad \gamma_m(t, A) = \int_0^t \int_A h(z, x(\tau-)) \mathbf{I}\{s(\tau-) = i\} \epsilon_z(dz) d\tau.$$

This simply implies that  $m(t, A) - \gamma_m(t, A)$  is a martingale.

Using the above notations, the evolution of the regime  $s(t)$  can be conveniently represented as

$$(2.3) \quad ds = \int_E \zeta(z) m(dt, dz)$$

with the compensator given by

$$(2.4) \quad \gamma_s(t) dt = \int_E \zeta(z) \gamma_m(dt, dz), \text{ where } \zeta(z) = \zeta((i, j)) = j - i$$

For example, if there is a regime shift from  $i$  to  $j$  occurred at time  $t$ , equation (2.3) then implies  $s_t = (j - i) + s_{t-} = j$ . Note that  $\int_E$  is equivalent to  $\sum_{i \neq j}$ .

**2.2. Two Factors.** Without loss of generality, we assume that the two factors determining the yield curve are characterized by the following stochastic differential equations<sup>4</sup>

$$(2.5) \quad dr_t = a(r_t, s_t) dt + b(r_t, s_t) dW_t$$

$$(2.6) \quad ds_t = \int_E \zeta(z) \mu(dt, dz)$$

where  $W_t$  is a standard Brownian motion,  $\mu(t, A)$  is the marked point process and they are independent.

**2.3. Pricing Kernel.** Absence of arbitrage allows us to specify the pricing kernel  $M_t$  as

$$(2.7) \quad \frac{dM_t}{M_t} = -r_t dt - \lambda_{D,t} dW_t - \int_E \lambda_S(z, r_{t-}) [\mu(dt, dz) - \gamma_\mu(dt, dz)]$$

where  $\lambda_{D,t} \equiv \lambda_D(r_t, s_t)$  is the market prices of diffusion risk; and  $\lambda_S(z, r_{t-})$  is the market price of regime switching (from regime  $i$  to regime  $j$ ) risk given  $r_{t-}$ . We assume that  $\lambda_D(r_t, s_t)$  and  $\lambda_S(z, r_{t-})$  are bounded in  $[0, T]$ .

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<sup>4</sup>We assume the stochastic differential equations have a unique solution and thus is well defined. The sufficient conditions can be found in Chapter 5 of Protter [25]. We further assume that  $r_t$  is bounded below.

Note that under the above assumptions, the explicit solution for  $M_t$  can be obtained by Doleans-Dade exponential formula (Protter [25]) as the following:

$$(2.8) \quad M_t = \exp \left\{ - \int_0^t r_u du \right\} \exp \left\{ - \int_0^t \lambda_{D,u} dW_u - \frac{1}{2} \int_0^t \lambda_{D,u}^2 du \right\} \times \\ \exp \left\{ \int_0^t \int_E \lambda_S(z, r_{u-}) \gamma_\mu(du, dz) + \int_0^t \int_E \log(1 - \lambda_S(z, r_{u-})) \mu(du, dz) \right\}$$

**2.4. The Term Structure of Interest Rates.** The specifications above completely pin down the yield curve. In this section we characterize the term structure of interest rates via the risk neutral probability measure. We first obtain the following two lemmas. The first lemma characterizes the equivalent martingale measure under which the interest rate term structure is determined. The second lemma specifies the dynamic of the short rate and the regime under the equivalent martingale measure.

LEMMA 2.1. *For fixed  $T > 0$ , the equivalent martingale measure  $\mathbf{Q}$  can be defined by the Radon-Nikodym derivative below*

$$\frac{d\mathbf{Q}}{dP} = \xi_T / \xi_0$$

where for  $t \in [0, T]$

$$(2.9) \quad \xi_t = \exp \left\{ - \int_0^t \lambda_{D,u} dW(u) - \frac{1}{2} \int_0^t \lambda_{D,u}^2 du \right\} \times \\ \exp \left\{ \int_0^t \int_E \lambda_S(z, r_{u-}) \gamma_\mu(du, dz) + \int_0^t \int_E \log(1 - \lambda_S(z, r_{u-})) \mu(du, dz) \right\}$$

provided  $E^P \{\xi_t\} = 1$  for  $t \in [0, T]$ .

PROOF. By Doleans-Dade exponential formula,  $\xi_t$  can be written in SDE form as

$$(2.10) \quad \frac{d\xi_t}{\xi_t} = -\lambda_{D,t} dW_t - \int_E \lambda_S(z, r_{t-}) [\mu(dt, dz) - \gamma_\mu(dt, dz)].$$

Since  $W_t$ , and  $\mu(t, A) - \gamma_\mu(t, A)$  are martingales under  $P$ ,  $\xi_t$  is a local martingale. Then, the assumption that  $E^P \{\xi_t\} = 1$  for  $t \in [0, T]$  ensures  $\xi_t$  is a martingale, and thus is the Radon-Nikodym derivative defining the equivalent martingale measure  $\mathbf{Q}$ .  $\square$

LEMMA 2.2. *Under the equivalent martingale measure  $\mathbf{Q}$ , the dynamics of  $r_t$  and  $s_t$  are given by the stochastic differential equations below*

$$(2.11) \quad dr_t = \tilde{a}(r_t, s_t) dt + b(r_t, s_t) d\tilde{W}_t$$

$$(2.12) \quad ds_t = \int_E \zeta(z) \tilde{\mu}(dt, dz)$$

where  $\tilde{a}(r_t, s_t) = a(r_t, s_t) - b(r_t, s_t) \cdot \lambda_D(r_t, s_t)$ ;  $\tilde{\mu}(t, A)$  is the marked point process with the stochastic intensity kernel as

$$\tilde{\gamma}_\mu(dt, dz) = \tilde{h}(z, r_{t-}) \mathbf{I}\{s_{t-} = i\} \epsilon_z(dz) dt,$$

where  $\tilde{h}(z, r_{t-}) = h(z, r_{t-})(1 - \lambda_S(z, r_{t-}))$  for all  $z \in E$ .

PROOF. Applying Girsanov's Theorem on the change of measure for Brownian motion, we have  $\tilde{W}_t = W_t - \int_0^t \lambda_D(r_u, s_u) du$  is a standard Brownian motion under  $\mathbf{Q}$ . This allows us to obtain  $\tilde{a}(r_t, s_t) = a(r_t, s_t) - b(r_t, s_t)\lambda_D(r_t, s_t)$ .

Since the marked point process,  $\mu(t, A)$ , is actually a collection of  $N(N - 1)$  conditional Poisson processes, by applying Girsanov's Theorem on conditional Poisson process (for example, see Theorem T2 and T3 in Chapter 6 of Bremaud [7]), the conditional Poisson process with intensity,  $h(z, r_{t-})$ , under  $\mathbf{P}$ , becomes the one with intensity,  $h(z, r_{t-})(1 - \lambda_S(z, r_{t-}))$  under  $\mathbf{Q}$ . Then, the result follows.  $\square$

To solve for the term structure of interest rates, note that, for fixed  $T$ , if  $P(t, T)$  is the price at time  $t$  of a riskless pure discount bond that matures at  $T$ , we must have, in the absence of arbitrage,

$$(2.13) \quad P(t, T) = E_t^{\mathbf{P}}(M_T/M_t) = E_t^{\mathbf{Q}} \exp\left\{-\int_t^T r_u du\right\}$$

with the boundary condition  $P(T, T) = 1$ .

Without loss of generality, let  $P(t, T) = f(t, r_t, s_t, T)$ . The following theorem gives the partial differential equation characterizing the bond price.

**THEOREM 2.3.** *In the setup of this section, the price of the risk-free pure discount bond  $f(t, r_t, s_t, T)$  defined in (2.13) satisfies the following system of partial differential equations*

$$(2.14) \quad f_t + \tilde{a}f_r + \frac{1}{2}b^2f_{rr} + \int_E \Delta_s f \tilde{h}\mathbf{I}\{s = i\}\epsilon_z(dz) = rf,$$

with the boundary condition:  $f(T, r, s, T) = 1$  for each  $s \in \{1, 2, \dots, N\}$ . Here  $f_t \equiv \frac{\partial f}{\partial t}$ ,  $f_r \equiv \frac{\partial f}{\partial r}$ ,  $f_{rr} \equiv \frac{\partial^2 f}{\partial r^2}$ , and  $\Delta_s f \equiv f(t, r_t, s_{t-} + \zeta(z), T) - f(t, r_t, s_{t-}, T)$ ,

PROOF. Applying Ito's formula for semimartingale (see Protter [25]) to  $f(t, r, s, T)$  under measure  $\mathbf{Q}$ ,

$$(2.15) \quad df = (f_t + \tilde{a}f_r + \frac{1}{2}b^2f_{rr})dt + bf_r d\tilde{W}_t + [f(t, r_t, s_t, T) - f(t, r_t, s_{t-}, T)]$$

Since the measures  $\mathbf{P}$  and  $\mathbf{Q}$  are equivalent, simultaneous jumps in  $\mu(t, A)$  is also of probability zero under  $\mathbf{Q}$ . Hence the last term in the above equation can be expressed by

$$\int_E \Delta_s f \tilde{\mu}(dt, dz).$$

Note that the term above can be made as martingales by subtracting its own compensator, which are added back in  $dt$  term. Therefore we have the following equation for  $f(t, r, s, T)$ :

$$(2.16) \quad \begin{aligned} df = & (f_t + \tilde{a}f_r + \frac{1}{2}b^2f_{rr} + \int_E \Delta_s f \tilde{h}\mathbf{I}\{s = i\}\epsilon_z(dz))dt \\ & + bf_r d\tilde{W}_t + \int_E \Delta_s f (\mu(dt, dz) - \tilde{\gamma}_\mu(dt, dz)) \end{aligned}$$

Since no arbitrage implies that the instantaneous expected returns of all assets should be the same as the short-term interest rate under the risk-neutral measure, equation (2.14) then follows by matching the coefficient of the  $dt$  term of (2.16) and  $rf$ .  $\square$

### 3. Two Tractable Specifications

In general equation (2.14) doesn't admit a closed form solution for the bond price. In this section, We consider two tractable specifications: affine and quadratic term structure of interest rates with regime switching.

**3.1. Affine Regime-Switching Models.** Duffie and Kan [12] and Dai and Singleton [9], among other, have detailed discussions of affine term structure models under diffusions. Duffie, Pan and Singleton [13] deals with general asset pricing under affine jump-diffusions. Bansal and Zhou [3] and Landen [22] all use affine structure for their regime switching models. Following this literature, we make the following parametric assumptions

ASSUMPTION 3.1. *The diffusion components of the short rate  $r_t$ , as well as those in the Markov switching process  $s_t$  all have an affine structure. In particular,*

- (1)  $a(r_t, s_t) = a_0(s_t) + a_1(s_t)r_t$ ,
- (2)  $b(r_t, s_t) = \sqrt{\sigma_0(s_t) + \sigma_1(s_t)r_t}$ ,
- (3)  $h(z, r_{t-}) = \exp\{h_0(z) + h_1(z)r_{t-}\}$ ;
- (4)  $\lambda_D(r_t, s_t) = \lambda_D(s_t)\sqrt{\sigma_0(s_t) + \sigma_1(s_t)r_t}$ ,
- (5)  $\lambda_S(z, r_t) = 1 - \exp\{\lambda_{0,S_t}(z) + \lambda_{1,S_t}(z)r_t\}$ .

The first three assumptions are related to the short rate process. We assume that the drift term and the volatility term of the diffusion part are all affine functions of  $r_t$  with regime-dependent coefficients. Then,  $r(t)$  becomes

$$(3.1) \quad dr = (a_0(s) + a_1(s)r) dt + \sqrt{\sigma_0(s) + \sigma_1(s)r} dB_t$$

We further assume that the log intensity of regime shifts is an affine function of the short term rate  $r_t$ . This assumption allow the transition probability to be time-varying.<sup>5</sup>

The last two assumptions deal with the market prices of risk. We assume that the market price of the diffusion risk is proportional to the volatility of the state variable  $r_t$  as in the conventional affine models as well as regime dependent. For the market price of regime switching risk, we assume that log of one minus market price of regime switching risk is affine of  $r_t$ . We pick this form because using log-linear approximation we may obtain a close-form solution to the bond pricing.

Under these parameterizations of the market prices of risk, the short rate  $r_t$  and the Markov chain  $s_t$  preserve the affine structure. In particular, under the risk-neutral measure  $\mathbf{Q}$  the drift term  $\tilde{a}(s, r)$ , and the *log* of regime switching intensity  $\tilde{h}(z, r)$  in (2.14) of Theorem 2.3 are affine functions of the instantaneous short-term interest  $r$  with state dependent coefficients:

$$\tilde{a}(s, r) = \tilde{a}_0(s) + \tilde{a}_1(s)r = a_0(s) - \lambda_D(s)\sigma_0(s) + [a_1(s) - \lambda_D(s)\sigma_1(s)]r$$

and

$$\begin{aligned} \tilde{h}(z, r) &= \exp\{\tilde{h}_0(z) + \tilde{h}_1(z)r\} \\ &= \exp\{(h_0(z) + \lambda_{0,S}(z)) + (h_1(z) + \lambda_{1,S}(z))r\}. \end{aligned}$$

Using a log-linear approximation similar to that in Bansal and Zhou [3], we can solve for the term structure of interest rates as follows:

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<sup>5</sup>Of course, a more general specification is to allow duration-dependence as well. However a closed-form solution for the yield curve may not be attainable. We are currently investigate this generalization.

**THEOREM 3.2.** *Under the Assumption 3.1, the price at time  $t$  of a risk-free pure discount bond with maturity  $\tau$  is given by  $P(s(t), r(t), \tau) = e^{A(\tau, s_t) + B(\tau, s_t)r_t}$  and the  $\tau$ -period interest rate is given by  $R(t, \tau) = -A(\tau, s_t)/\tau - B(\tau, s_t)r_t/\tau$ , where  $A(\tau, s)$  and  $B(\tau, s)$  are determined by the following differential equations*

$$(3.2) \quad \begin{aligned} & -\frac{\partial B(\tau, s)}{\partial \tau} + \tilde{a}_1(s)B(\tau, s) + \frac{1}{2}\sigma_1(s)B^2(\tau, s) \\ & + \int_E \left[ e^{\Delta_s A} (\Delta_s B + \tilde{h}_1(z)) - \tilde{h}_1(z) \right] e^{\tilde{h}_0(z)} \mathbf{1}(s=i) \epsilon_z(dz) = 1 \end{aligned}$$

and

$$(3.3) \quad \begin{aligned} & -\frac{\partial A(\tau, s)}{\partial \tau} + \tilde{a}_0(s)B(\tau, s) + \frac{1}{2}\sigma_0(s)B^2(\tau, s) \\ & + \int_E \left[ e^{\Delta_s A} - 1 \right] e^{\tilde{h}_0(z)} \mathbf{1}(s=i) \epsilon_z(dz) = 0 \end{aligned}$$

with boundary conditions  $A(0, s) = 0$  and  $B(0, s) = 0$ , where  $\Delta_s A = A(\tau, s + \zeta(z)) - A(\tau, s)$  and  $\Delta_s B = B(\tau, s + \zeta(z)) - B(\tau, s)$

**PROOF.** Without loss of generality, let the price at time  $t$  of a pure-discount bond that will mature at  $T$  be given as

$$f(t, s(t), x(t), T) = P(s(t), r(t), \tau) = e^{A(\tau, s(t)) + B(\tau, s(t))r(t)}$$

where  $\tau = T - t$  and  $A(0, s) = 0$ ,  $B(0, s) = 0$ .

Theorem 2.3 then implies

$$(3.4) \quad \begin{aligned} r &= -\frac{\partial A(\tau, s)}{\partial \tau} - \frac{\partial B(\tau, s)}{\partial \tau} r \\ &+ [\tilde{a}_0(s) + \tilde{a}_1(s)r]B(\tau, s) + \frac{1}{2}[\sigma_0(s) + \sigma_1(s)r]B^2(\tau, s) \\ &+ \int_E (e^{\Delta_s A + \Delta_s B r} - 1) e^{\tilde{h}_0(z) + \tilde{h}_1(z)r} \mathbf{1}(s=i) \epsilon_z(dz) \end{aligned}$$

where  $\Delta_s A = A(\tau, s + \zeta(z)) - A(\tau, s)$  and  $\Delta_s B = B(\tau, s + \zeta(z)) - B(\tau, s)$

Using the log-linear approximation,

$$e^{(\Delta_s B + \tilde{h}_1)r} \approx 1 + (\Delta_s B + \tilde{h}_1)r, \text{ and } e^{\tilde{h}_1 r} \approx 1 + \tilde{h}_1 r,$$

we have

$$(3.5) \quad \begin{aligned} & (e^{\Delta_s A + \Delta_s B r} - 1) e^{\tilde{h}_0 + \tilde{h}_1 r} \\ &= e^{(\Delta_s A + \tilde{h}_0) + (\Delta_s B + \tilde{h}_1)r} - e^{\tilde{h}_0 + \tilde{h}_1 r} \\ &\approx e^{(\Delta_s A + \tilde{h}_0)} \left( 1 + (\Delta_s B + \tilde{h}_1)r \right) - e^{\tilde{h}_0} \left( 1 + \tilde{h}_1 r \right) \\ &= e^{\tilde{h}_0} \left( e^{\Delta_s A} - 1 \right) + \left[ e^{\Delta_s A + \tilde{h}_0} (\Delta_s B + \tilde{h}_1) - e^{\tilde{h}_0} \tilde{h}_1 \right] r. \end{aligned}$$

Theorem 3.1 follows by substituting the above approximation into (3.4) and matching the coefficients on  $r$  on both side of the equation.  $\square$

Note that Theorem 3.1 includes the affine model by Duffie and Kan [12] Bansal and Zhou [3] and Landen [22] as special cases. When there is only one regime,  $\Delta_s B = 0$  and  $e^{\Delta_s A} - 1 = 0$ . Then, (3.2) and (3.3) reduce to the equations for bond pricing of affine models. Without using the log-linear approximation, Landen [22] only considers models where  $\Delta_s B = 0$  and is silent on the market price of

regime switching risk. In the case of Bansal and Zhou [3], the risk of regime shifts is not priced neither, i.e.  $\tilde{h}_0(z) = h_0(z)$ . The model in Theorem 3.2 is in fact a special case of that in Dai and Singleton [10], which proposes a general dynamic term structure model where the risk of regime shifts is priced. The main difference between the current paper and Dai and Singleton [10] is that we also provided an explicit solution for the term structure of interest rates using log-linear approximation even when both the coefficients of drift and diffusion are regime-dependent.

To better see the difference between the model in Bansal and Zhou [3] and the current one, we can examine the expected excess return on a long term bond over the short rate implied by our model. Consider a long term bond with maturity  $\tau$  whose price is given by  $P(t, \tau) = e^{A(\tau, s_t) + B(\tau, s_t)r_t}$ . Using Ito's formula, we can easily obtain

$$(3.6) \quad E_t \left( \frac{dP_t}{P_t} \right) - r_t = \lambda_D(s_t, r_t) b(s_t, r_t) B(\tau, s_t) + \int (e^{\Delta_s A + \Delta_s B \cdot r_t} - 1) h(z, r_t) \lambda_S(z, r_t) \mathbf{1}(s_t = i) \epsilon_z(dz)$$

The first term on the right hand side of equation (3.6) is interpreted as the diffusion risk premium in the literature, and the second can be analogously defined as the regime-switching risk premium. The equation shows that introducing the dependence of the market prices of diffusion on  $s_t$  adds more flexibility to the specification of the risk premium. Bansal and Zhou [3] points out that it is mainly this feature of the regime switching model that provides improved goodness-of-fit over the existing term structure models. On the other hand, (3.6) also shows that if the term structure exhibits significant difference across regimes ( $\Delta_s A \neq 0$  or  $\Delta_s B \neq 0$ ), there is an additional source of risk due to regime shifts and it should also be priced ( $\lambda_S(z, r_t)$ ) in the term structure model. Introducing the regime switching risk not only can add more flexibilities to the specification of time-varying bond risk premiums, but also can be potentially important in understanding the bond risk premiums over different holding periods. Wu and Zeng [27] use a general equilibrium model to introduce the systematic risk of regime shift in the term structure of interest rate and further estimate the model by Efficient Method of Moments. They find that the market price of the regime-switching risk is not only statistically significant, but also economically important, accounting for a significant portion of the term premiums for long-term bonds. Ignoring the regime-switching risk leads to underestimation of long-term interest rates and therefore flatter yield curves.

**3.2. Quadratic Regime-Switching Models.** Quadratic term structure models for interest rate also offer tractable structures and is another class of useful models. Ahn et.al. [1] is a recent paper provides theory and empirical evidence on the quadratic term structure models. Here, we incorporate regime-switching into them.

In the quadratic case, we make the following assumptions for the short rate and the Markov switching process.

**ASSUMPTION 3.3.** *The diffusion components of the short rate  $r_t$ , as well as those in the Markov switching process  $s_t$  have the following structure:*

$$(1) \quad a(r_t, s_t) = a_0(s_t) + a_1(s_t)r_t,$$



- (2)  $b(s_t) = \sqrt{\sigma(s_t)}$ ,  
 (3)  $h(z, r_{t-}) = e^{h_0(z) + h_1(z)r_{t-} + h_2(z)r_{t-}^2}$ ;  
 (4)  $\lambda_D(s_t) = \lambda_D(s_t)\sqrt{\sigma_0(s_t)}$ ,  
 (5)  $\lambda_S(z, r_t) = 1 - e^{\lambda_{0,S}(z) + \lambda_{1,S}(z)r_t + \lambda_{2,S}(z)r_t^2}$ .

The first two assumptions imply the short rate process is a Gaussian process, which is Vasicek's model with regime dependent coefficients. The third assumption specifies that the log intensity of regime shifts is a quadratic function of the short term rate  $r_t$ . The last two assumptions deal with the market prices of risk. We assume that the market prices of the diffusion risk are regime-dependent constants. For the market price of regime switching risk, we assume that log of one minus market price of regime switching risk is a quadratic function of  $r_t$ . We pick this form because using log-linear approximation we may obtain a close-form solution to the bond pricing. Here,  $b$  and  $\lambda_D$  can only be constants. If they have linear or quadratic terms, their coefficients will be zero in the coefficient matching.

Under the risk-neutral measure  $\mathbf{Q}$  the drift term  $\tilde{a}(s, r)$ , and the *log* of regime switching intensity  $\tilde{h}(z, r)$  in (2.14) of Theorem 2.3 become

$$\tilde{a}(s, r) = \tilde{a}_0(s) + \tilde{a}_1(s)r = a_0(s) - \lambda_D(s)\sigma(s) + a_1(s)r$$

and

$$\begin{aligned} \tilde{h}(z, r) &= \exp\{\tilde{h}_0(z) + \tilde{h}_1(z)r + \tilde{h}_2r^2\} \\ &= \exp\{(h_0(z) + \lambda_{0,S}(z)) + (h_1(z) + \lambda_{1,S}(z))r + (h_2(z) + \lambda_{2,S}(z))r^2\}. \end{aligned}$$

Similarly, using a log-linear approximation, we can solve for the term structure of interest rates in quadratic form as follows:

**THEOREM 3.4.** *Under the Assumption 3.3, the price at time  $t$  of a risk-free pure discount bond with maturity  $\tau$  is given by  $P(s(t), r(t), \tau) = e^{A(\tau, s_t) + B(\tau, s_t)r_t + C(\tau, s_t)r_t^2}$  and the  $\tau$ -period interest rate is given by  $R(t, \tau) = -(A(\tau, s_t) - B(\tau, s_t)r_t - C(\tau, s_t)r_t^2)/\tau$ , where  $A(\tau, s)$ ,  $B(\tau, s)$  and  $C(\tau, s)$  are determined by the following differential equations*

$$(3.7) \quad \begin{aligned} & -\frac{\partial C(\tau, s)}{\partial \tau} + 2\tilde{a}_1(s)C(\tau, s) + 2\sigma(s)C^2(\tau, s) \\ & + \int_E \left[ e^{\Delta_s A} (\Delta_s C + \tilde{h}_2(z)) - \tilde{h}_2(z) \right] e^{\tilde{h}_0(z)} \mathbf{1}(s=i) \epsilon_z(dz) = 0 \end{aligned}$$

$$(3.8) \quad \begin{aligned} & -\frac{\partial B(\tau, s)}{\partial \tau} + 2\tilde{a}_0C(\tau, s) + \tilde{a}_1(s)B(\tau, s) + 2\sigma(s)B(\tau, s)C(\tau, s) \\ & + \int_E \left[ e^{\Delta_s A} (\Delta_s B + \tilde{h}_1(z)) - \tilde{h}_1(z) \right] e^{\tilde{h}_0(z)} \mathbf{1}(s=i) \epsilon_z(dz) = 1 \end{aligned}$$

and

$$(3.9) \quad \begin{aligned} & -\frac{\partial A(\tau, s)}{\partial \tau} + \tilde{a}_0(s)B(\tau, s) + \frac{1}{2}\sigma(s)(2C(\tau, s) + B^2(\tau, s)) \\ & + \int_E [e^{\Delta_s A} - 1] e^{\tilde{h}_0(z)} \mathbf{1}(s=i) \epsilon_z(dz) = 0 \end{aligned}$$

with boundary conditions  $A(0, s) = B(0, s) = C(0, s) = 0$ , and  $\Delta_s A = A(\tau, s + \zeta(z)) - A(\tau, s)$ ,  $\Delta_s B = B(\tau, s + \zeta(z)) - B(\tau, s)$  and  $\Delta_s C = C(\tau, s + \zeta(z)) - C(\tau, s)$ .

PROOF. Without loss of generality, let the price at time  $t$  of a pure-discount bond that will mature at  $T$  be given as

$$f(t, s(t), x(t), T) = P(s(t), r(t), \tau) = e^{A(\tau, s(t)) + B(\tau, s(t))r(t) + C(\tau, s(t))r^2(t)}$$

where  $\tau = T - t$  and  $A(0, s) = B(0, s) = C(0, s) = 0$ .

Theorem 2.3 then implies

$$(3.10) \quad \begin{aligned} r = & -\frac{\partial A(\tau, s)}{\partial \tau} - \frac{\partial B(\tau, s)}{\partial \tau}r - \frac{\partial C(\tau, s)}{\partial \tau}r^2 \\ & + [\tilde{a}_0(s) + \tilde{a}_1(s)r](B(\tau, s) + 2C(\tau, s)r) \\ & + \frac{1}{2}\sigma(s)\left(2C(\tau, s) + (B(\tau, s) + 2C(\tau, s)r)^2\right) \\ & + \int_E \left(e^{\Delta_s A + \Delta_s B r + \Delta_s C r^2} - 1\right) e^{\tilde{h}_0(z) + \tilde{h}_1(z)r + \tilde{h}_2(z)r^2} \mathbf{1}(s = i) \epsilon_z(dz) \end{aligned}$$

where  $\Delta_s A = A(\tau, s + \zeta(z)) - A(\tau, s)$ ,  $\Delta_s B = B(\tau, s + \zeta(z)) - B(\tau, s)$  and  $\Delta_s C = C(\tau, s + \zeta(z)) - C(\tau, s)$ .

Using the log-linear approximation,

$$e^{(\Delta_s B + \tilde{h}_1)r + (\Delta_s C + \tilde{h}_2)r^2} \approx 1 + (\Delta_s B + \tilde{h}_1)r + (\Delta_s C + \tilde{h}_2)r^2,$$

and

$$e^{\tilde{h}_1 r + \tilde{h}_2 r^2} \approx 1 + \tilde{h}_1 r + \tilde{h}_2 r^2$$

we have

$$(3.11) \quad \begin{aligned} & \left(e^{\Delta_s A + \Delta_s B r + \Delta_s C r^2} - 1\right) e^{\tilde{h}_0 + \tilde{h}_1 r + \tilde{h}_2(z)r^2} \\ & = e^{(\Delta_s A + \tilde{h}_0) + (\Delta_s B + \tilde{h}_1)r + (\Delta_s C + \tilde{h}_2)r^2} - e^{\tilde{h}_0 + \tilde{h}_1 r + \tilde{h}_2 r^2} \\ & \approx e^{(\Delta_s A + \tilde{h}_0)} \left(1 + (\Delta_s B + \tilde{h}_1)r + (\Delta_s C + \tilde{h}_2)r^2\right) - e^{\tilde{h}_0} \left(1 + \tilde{h}_1 r + \tilde{h}_2 r^2\right) \\ & = e^{\tilde{h}_0} \left(e^{\Delta_s A} - 1\right) + \left[e^{\Delta_s A + \tilde{h}_0} (\Delta_s B + \tilde{h}_1) - e^{\tilde{h}_0} \tilde{h}_1\right] r \\ & \quad + \left[e^{\Delta_s A + \tilde{h}_0} (\Delta_s C + \tilde{h}_2) - e^{\tilde{h}_0} \tilde{h}_2\right] r^2. \end{aligned}$$

Theorem 3.4 follows by substituting the above approximation into (3.10) and matching the coefficients on  $r$  on both side of the equation.  $\square$

#### 4. Conclusion

The regimes underlying the term structure of interest rates are shown to be closely related to business cycle fluctuations in the previous studies. Thus the risk of regime shifts is very likely to be a systematic risk. The term structure models developed in the current paper offer flexible parameterizations of the market price of regime-switching risk. Closed-form solutions for the term structure of interest rates are obtained for both affine- and quadratic-type models using log-linear approximations. Such systematic risk of regime shifts is also likely to have important implications for pricing interest rate derivatives (e.g. Singleton and Umantsev [26]) as well as for investors' optimal portfolio choice problem (e.g. Campbell and Viceira [8]). Moreover, motivated by the observation of persistent monetary policy actions and their impact on interest rates, the models can be extended to the framework of affine regime-switching jump diffusion for term structure of interest rates. These extensions are left for future research.

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