An Econometric Model of The Term Structure of Interest Rates Under Regime-Switching Risk

Shu Wu and Yong Zeng

Abstract This paper develops and estimates a continuous-time model of the term structure of interests under regime shifts. The model uses an analytically simple representation of Markov regime shifts that elucidates the effects of regime shifts on the yield curve and gives a clear interpretation of regime-switching risk premiums. The model falls within the broad class of essentially affine models with a closed form solution of the yield curve, yet it is flexible enough to accommodate priced regime-switching risk, time-varying transition probabilities, regime-dependent mean reversion coefficients as well as stochastic volatilities within each regime. A two-factor version of the model is implemented using Efficient Method of Moments. Empirical results show that the model can account for many salient features of the yield curve in the U.S.

1 Introduction

Economic theories relate asset prices, and interest rates in particular, to either observable or latent variables that summarize the state of the aggregate economy. Since one important characteristic of the aggregate economy is the recurrent shifts between distinct phases of the business cycle, economists have long used models that incorporate Markov regime shifts to describe the stochastic behavior of interest rates. Some examples include Hamilton [33], Lewis [40], Cecchetti et al. [10], Sola and Driffill [45], Garcia and Perron [31], Gray [32] and Ang and Bekaert [2] among others. Typically these studies model the short-term interest rate as a stochastic pro-

Shu Wu

Department of Economics, University of Kansas, 1450 Jayhawk Blvd, Lawrence, KS 66045, United States. e-mail: shuwu@ku.edu

Yong Zeng

Department of Mathematics and Statistics, University of Missouri at Kansas City, 5100 Rockhill Rd, KCMO, Missouri 64110, United States. e-mail: zengy@umkc.edu

cess with time-varying parameters that are driven by a Markov state variable. Long-term interest rates can then related to the short rate through the expectation hypothesis. Results from these studies suggest that regime-switching models in general have better empirical performance than their single-regime counterparts. Regime-switching models are shown to be able to capture the non-linearities in the drift and volatility function of the short rate found in non-parametric models (Ang and Bekaert[3]).

The success of these empirical studies have motivated a growing literature that examine the impact of regime shifts on the entire yield curve using dynamic term structure models. For example, Boudoukh et al. [9] investigates the implications of a 2-regime model of the business cycle based on GDP, consumption and production data for term premiums and volatilities in the bond market. Bansal and Zhou [5] and Bansal et al. [6] incorporate a Markov-switching state variable into the parameters of an otherwise standard multi-factor Cox-Ingersoll-Ross (CIR) model of the term structure of interest rates. A closed-form solution for the yield curve is obtained under log-linear approximation. They find that the key to the better empirical performance of the regime-switching model is the added flexibility of the market price of risk under multiple regimes, and regimes in the term structure model are intimately related to bond risk premiums and the business cycle. Evans [23] develops and estimates a dynamic term structure model under regime shifts for both nominal and real interest rates in Britain. In a similar study, Ang et al. [4] also develops a non-arbitrage regime-switching model of the term structure of interest rates with both nominal bond yields and inflation data to efficiently identify the term structure of real rates and inflation risk premia. Different from the model in Evans [23], Ang et al. [4] allows inflation and real rates to be driven by two different regime variables. Dai et al. [16] emphasizes that not only regime shifts can affect parameters of the state variables, but also regime-switching risk should be priced in dynamic term structure models. Using monthly data on the U.S. Treasury zero-coupon bond yields, they show that the priced regime-switching risk plays a critical role in capturing the time variations in the expected excess bond returns. These studies all use discrete-time models. More recent examples include Ferland et al. [26], Futami [27] and Xiang and Zhu [48] among others.

In this paper we contribute to this literature by developing and estimating a continuous-time model of the term structure of interests under regime shifts. Most of the existing regime-switching models are specified in a discrete-time framework. Compared to those models, our continuous-time model has several advantages. (1) It uses an analytically simple representation of Markov regime shifts that helps elucidate the effect of regime shifts on the yield curve; (2) It offers a clear economic interpretation of the market price of regime-switching risk; (3) It gives a tractable solution of the term structure of interest rates in the presence of time-varying transition probabilities, regime-dependent mean reversion coefficients, priced regime-switching risk, and stochastic volatilities conditional on each regime without using log-linear approximations; (4) A continuous-time model is also more convenient in the applications of the pricing of interest rate derivatives.

The model presented in this paper falls within the broad class of affine models of Duffie and Kan [18], Dai and Singleton [13], Duffie (2002), and more recently Aït-Sahalia and Kimmel [1] and Le et al. [39]. The model implies that bond risk premiums include two components under regime shifts. One is a regime-dependent risk premium due to diffusion risk. The other is a regime-switching risk premium that depends on the covariations between the discrete changes in bond prices and the stochastic discount factor across different regimes. This new component of the term premiums is associated with the systematic risk of recurrent shifts in bond prices (or interest rates) due to regime changes and is an important factor that affects bond returns.

One stylized fact about the term structure of interest rates is that long-term rates do not attenuate in volatility. In the standard affine models, volatility of interest rates depends the factor loadings, which can converge to zero quickly unless the underlying state variables are very persistent (under the risk-neutral probability measure). The model in the present paper shows that regime shifts introduce an additional source of volatility that can equally affect both the short and the long end of the yield curve. Therefore the model is able to generate volatile long-term interest rates even when the underlying state variables are not very persistent.

Other continuous-time term structure models under regime shifts include Landen [37] which uses a similar representation for Markov regime shifts as that in the current paper. Landen [37], however, only solves the term structure of interest rates under the risk-neutral probability measure, and is silent on the market price of risk. Dai and Singleton [15] also proposes a continuous-time model of the term structure of interest rates under regime shifts based on a different representation of Markov regime shifts. Both studies did not implement their models empirically. Wu and Zeng [47] develops a general equilibrium model of the term structure of interest rates under regime shifts similar to that in Cox et al. [12]. The focus of their study is on the general equilibrium interpretation of the regime-switching risk. And unlike the present paper, they obtain the solution of the yield curve under log-linear approximations following Bansal and Zhou [5]. Separately, exponential affine models of bond prices under regime switching are also derived in Elliott and Siu [21] and Siu [43].

The rest of the paper is organized as follows. Section 2 presents the theoretical model and examines the effects of regime shifts on the term structure of interest rates. A closed-form solution of the term structure of interest rates is obtained for an essentially affine model. Section 3 implements a two-factor version of the model using Efficient Method of Moments and discusses the empirical results. Section 4 contains some concluding remarks and possible extensions of the model.

2 The model

2.1 A simple representation of Markov regime shifts

In order to obtain a simple and closed-form solution of the yield curve, we first show that Markov regime shifts can be modeled as a marked point process.[1] The main advantage of this new representation of regime shifts is that it allows us to elucidates the role of regime shifts in determining the term structure of interest rates with a clear interpretation of the regime-switching risk premiums.

We assume that there are N possible regimes and denote $S(t)$ as the regime at time t. Let the mark space $U = \{1, 2, ..., N\}$ be all possible regimes with the power σ-algebra. We denote u as a generic point in U and A as a subset of U. A marked point process or a random counting measure, $m(t, A)$, is defined as the total number of times we enter a regime that belongs to A during $(0, t]$. For example, $m(t, \{u\})$ simply counts the total number of times we enter regime u during $(0, t]$. We also define η as the usual counting measure on U with the following two properties: For $A \in U$, $\eta(A) = \int I_A \eta(du)$ (i.e. $\eta(A)$ counts the number of elements in A) and $\int_A f(u)\eta(du) = \sum_{u \in A} f(u)$.

The probability laws of the marked point process defined above, $m(t, \cdot)$, can be uniquely characterized by a stochastic intensity kernel,[2] which is assumed to be

$$\gamma_m(dt, du) = h(u; S(t-), X(t))\eta(du)dt, \tag{1}$$

where $X(t)$ is a vector of other continuous state variables to be specified below; $h(u, S(t-), X(t))$ is the conditional regime-shift (from regime $S(t-)$ to u) intensity at time t (we assume $h(u, S(t-), X(t))$ is bounded) that is measurable with respect to u, $S(t-)$ and $X(t)$. The $N \times N$ conditional intensity matrix of regime-switching is $H(X(t)) = \{h(j, i, X(t))\}$ with $h(i, j, X(t)) = 0$ when $i = j$. Heuristically, $\gamma_m(dt, du)$ can be thought of as the (time-varying) conditional probability of shifting from Regime $S(t-)$ to Regime u during $[t-, t+dt)$ given $X(t)$ and $S(t-)$. Note that $\gamma_m(t, A)$, the compensator of $m(t, A)$,[3] can be written as

$$\gamma_m(t, A) = \int_0^t \int_A h(u, S(\tau-), X(\tau))\eta(du)d\tau = \sum_{u \in A} \int_0^t h(u, S(\tau-), X(\tau))d\tau.$$

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[1] In the context of continuous-time models, Landen [37] also uses a marked point process to represent Markov regime shifts in her model of the term structure of interest rates. However, we use a different construction of the mark space that simplifies the corresponding random measure. Other approaches to regime shifts include Hidden Markov Models (e.g. Elliott et al. [19]) and the Conditional Markov Chain models (e.g. Yin and Zhang [49]). An application of Hidden Markov Models to the term structure of interest rates can be found in Elliott and Mamon [20]. Bielecki and Rutkowski [7, 8] are examples of the application of conditional Markov Chain models to the term structure of interest rates.

[2] See Last and Brandt [38] for detailed discussion of marked point process, stochastic intensity kernel and related results.

[3] This simply means that $m(t, A) - \gamma_m(t, A)$ is a martingale.

With the market point process appropriately defined, we now can represent the regime, $S(t)$, as an integral along $m(\cdot, \cdot)$ as that in He et al. [36],

$$S(t) = S(0) + \int_{[0,t] \times U} (u - S(\tau-))m(d\tau, du). \tag{2}$$

Note that $m(d\tau, du)$ is 0 most of time and only becomes 1 at a regime-switching time t_i with $u = S(t_i)$, the new regime at time t_i. In other words, the above expression is equivalent to a telescoping sum: $S(t) = S(0) + \sum_{t_i < t}(S(t_i) - S(t_{i-1}))$.

We can also describe the evolution of regimes $S(t)$ in a differential form

$$dS(t) = \int_U (u - S(t-))m(dt, du). \tag{3}$$

To see the above differential equation is valid, assuming there is a regime shift from $S(t-)$ to u at time t, then $S(t) - S(t-) = (u - S(t-))$, implying $S(t) = u$.

Alternatively, we can express $dS(t)$ as

$$dS(t) = \int_U (u - S(t-))\gamma_m(dt, du) + \int_U (u - S(t-))[m(dt, du) - \gamma_m(dt, du)]. \tag{4}$$

where $\gamma_m(dt, du)$ is the intensity kernel of $m(dt, du)$. And by construction $m(dt, du) - \gamma_m(dt, du)$ is a matingale error term, and hence can be thought of as a *regime-switching shock* whereas the first term is the conditional expectation of $dS(t)$.[4]

2.2 Other state variables

We assume that, in addition to the Markov regime-switching variable $S(t)$, there are L other *continuous* state variables represented by a $L \times 1$ vector $X(t)$. Without loss of generality we assume that $X(t)$ is given by the following stochastic differential equation

$$dX(t) = \Theta(X(t), S(t-))dt + \Sigma(X(t), S(t-))dW(t) \tag{5}$$

where $\Theta(X, S)$ is a $L \times 1$ vector; $\Sigma(X, S)$ is a $L \times L$ matrix; $W(t)$ is a $L \times 1$ vector of standard Brownian motions that is independent of S(t). Note that in this specification, both the drift term $\Theta(\cdot, \cdot)$ and the diffusion term $\Sigma(\cdot, \cdot)$ are regime dependent. This general specification also allows stochastic volatility within each regime. The time-path of $X(t)$, however, is continuous.

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[4] This is analogous to the representation of Markov regime shifts as an AR(1) process in discrete-time models.

2.3 The term structure of interest rates

Let $M(t)$ denote the pricing kernel.[5] We assume that $M(t)$ is given by

$$
\begin{aligned}
\frac{dM(t)}{M(t-)} = & -r(t-)dt - \lambda'_D(X(t),S(t-))dW(t) \\
& - \int_U \lambda_S(u,S(t-),X(t))[m(dt,du) - \gamma_m(dt,du)]
\end{aligned}
\tag{6}
$$

where $r(t)$ is the instantaneous short-term interest rate, $\lambda_D(X,S)$ is a $L \times 1$ vector of market prices of diffusion risk, and $\lambda_S(u,S,X)$ is the market price of regime-switching (from regime $S(t-)$ to regime u) risk given X_t. The interpretations for λ_D and λ_S will become much clearer from the discussions below.

Note that the explicit solution for $M(t)$ can be obtained by Doleans-Dade exponential formula (Protter [41]) as follows

$$
M(t) = \left(e^{-\int_0^t r_{\tau-}d\tau} \right) \left(e^{-\int_0^t \lambda'_D(X_\tau,S_{\tau-})dW(\tau) - \frac{1}{2}\int_0^t \lambda'_D(X_\tau,S_{t-})\lambda_D(X_\tau,S_{\tau-})d\tau} \right) \times \\
\left(e^{\int_0^t \int_U \lambda_S(u,S_{\tau-},X_\tau)\gamma_m(d\tau,du) + \int_0^t \int_U \log(1-\lambda_S)(u,S_{\tau-},X_\tau)m(d\tau,du)} \right)
\tag{7}
$$

The term structure of interest rates can be obtained by a change of probability measure. We first obtain the following two lemmas. The first lemma characterizes the equivalent martingale measure under which the yield curve is determined. The second lemma obtains the dynamic of the state variables under the equivalent martingale measure.

Lemma 1. *For fixed $T > 0$, the equivalent martingale measure* **Q** *can be defined by the Radon-Nikodym derivative below*

$$
\frac{dQ}{dP} = \xi(T)/\xi_0
$$

where for $t \in [0,T]$

$$
\xi(t) = \left(e^{-\int_0^t \lambda'_D(X_\tau,S_{\tau-})dW(\tau) - \frac{1}{2}\int_0^t \lambda'_D(X_\tau,S_{\tau-})\lambda_D(X_\tau,S_{\tau-})d\tau} \right) \times \\
\left(e^{\int_0^t \int_U \lambda_S(u,S_{\tau-},X_\tau)\gamma_m(d\tau,du) + \int_0^t \int_U \log(1-\lambda_S(u,S_{\tau-},X_\tau)m(d\tau,du)} \right)
\tag{8}
$$

provided λ_D satisfies Kazamaki or Novikov's criterion and λ_S and h (the stochastic intensity kernel of $m(t,A)$) are all bounded functions.

Lemma 2. *Under the risk-neutral probability measure* **Q***, the dynamics of state variables, $X(t)$ and $S(t)$, are given by the following stochastic differential equations respectively*

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[5] Absence of arbitrage is sufficient for the existence of the pricing kernel under certain technical conditions, as pointed out by Harrison and Kreps [35]

$$dX(t) = \tilde{\Theta}(X(t), S(t-))dt + \Sigma(X(t), S(t-))d\tilde{W}(t) \qquad (9)$$

$$dS(t) = \int_U (u - S(t-))\tilde{m}(dt, du) \qquad (10)$$

where $\tilde{\Theta}(X, S) = \Theta(X, S) - \Sigma(X, S)\lambda_D(X, S)$, $\tilde{W}(t)$ is a $L \times 1$ standard Brownian motion and $\tilde{m}(t, A)$ is a marked point process with intensity matrix $\tilde{H}(X) = \{\tilde{h}(j, i, X)\} = \{h(j, i, X)(1 - \lambda_S(j, i, X))\}$, under $\mathbf{Q}$, respectively.

Note that the compensator of $\tilde{m}(t, A)$ under $\mathbf{Q}$ becomes

$$\tilde{\gamma}_m(dt, du) = \tilde{h}(u, S(t-), X(t))\eta(du)dt = (1 - \lambda_S(u, S(t-), X(t)))\gamma_m(dt, du).$$

In the absence of arbitrage, the price at time $t-$ of a default-free pure discount bond that matures at T, $P(t-, T)$, can be obtained as

$$P(t-, T) = E^{\mathbf{Q}}\left[e^{-\int_t^T r_\tau - d\tau}|\mathcal{F}_{t-}\right] = E^{\mathbf{Q}}\left[e^{-\int_t^T r_\tau - d\tau}|X(t), S(t-)\right] \qquad (11)$$

with the boundary condition $P(T, T) = 1$. The last equality comes from the Markov property of $(X(t), S(t))$. Without loss of generality, let $P(t-, T) = f(t, X(t), S(t-), T)$. The following proposition gives the partial differential equation determining the bond price.

Proposition 1. *The price of the default-free pure discount bond $f(t, X, S, T)$ defined in (11) satisfies the following partial differential equation*

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X'}\tilde{\Theta} + \frac{1}{2} tr\left(\frac{\partial^2 f}{\partial X \partial X'}\Sigma\Sigma'\right) + \int_U \Delta_S f\, \tilde{h}(u, S, X)\eta(du) = rf \qquad (12)$$

with the boundary condition $f(T, X, S, T) = 1$, where $\Delta_S f \equiv f(t, X, u, T) - f(t, X, S, T)$.

2.4 Bond risk premiums under regime shifts

In general equation (12) doesn't admit a closed form solution for the bond price. Nonetheless, the equation allows us to illustrate how regime shifts affect bond risk premiums and give a clear interpretation the market price of regime-switching risk, λ_S.

By Ito's formula, we have

$$df = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X'}\theta(X, S) + \frac{1}{2}tr\left(\frac{\partial^2 f}{\partial X \partial X'}\Sigma(X, S)\Sigma'(X, S)\right)\right]dt + \frac{\partial f}{\partial X'}\Sigma(X, S)dW$$

$$+ \int_U \Delta_S f\, \gamma_m(dt, du) + \int_U \Delta_S f\,(m(dt, du) - \gamma_m(dt, du))$$

$$(13)$$

Using (12), Lemma 2 as well as the definition of $\gamma_m(dt, du)$ in Equation (1), we can easily obtain

$$
\begin{aligned}
E_{t-}\left(\frac{df}{f}\right) - rdt &= \left[\frac{1}{f}\frac{\partial f}{\partial X'}\Sigma(X,S)\lambda_D(X,S)\right]dt \\
&+ \left[\int_U \frac{\Delta_S f}{f}\lambda_S(u,X,S)h(u,X,S)\eta(du)\right]dt
\end{aligned}
\tag{14}
$$

The left-hand side of (14) gives the (instantaneous) expected excess return on a zero-coupon bond, or risk premium. The equation shows that the bond risk premium includes two components under regime shifts. Conditional on regime S, the first term is the minus of the covariance between the bond return, df/f, and the change in the pricing kernel, dM/M, which can be interpreted as investors' marginal utility growth, due to shocks $dW(t)$ (see Equation (6) for the specification of $M(t)$). We refer to this term in this paper as the diffusion risk premium. This risk premium is in general time-varying due to the presence of $X(t)$ and $S(t)$ in Σ and λ_D.[6] Equation (14) shows clearly that, compared to single-regime models in which the diffusion risk premium depends only on $X(t)$, regime shifts introduces an additional source of time variation in the risk premiums as $S(t)$ changes randomly over time. Since researchers often attribute the failure of the expectation theory of the term structure of interest rates to time-varying risk premiums, regime shifts therefore can potentially improve the empirical performance of dynamic term structure models. In fact, Bansal and Zhou [5] argues that the regime-dependence of the diffusion risk premium plays a crucial role in enabling their econometric model to account for the failure of the expectation theory.

Equation (14) also makes it clear that, under regime shifts, bond risk premiums in general include a second component. To understand more clearly what the second component is about, recall that $-\lambda_S$ simply gives the impact of a regime-switching shock $m(dt, du) - \gamma_m(dt, du)$ on dM/M in Equation (6), whereas $\frac{\Delta_S f}{f}$ has a similar interpretation in Equation (13). Also recall that $h(u, X, S)$ is the regime-switching intensity (from S to u). Therefore $\int_U \frac{\Delta_S f}{f}\lambda_S(X,S)h(u,X,S)\eta(du)$ is again the minus of the *covariance* between the bond return, df/f, and the change of the pricing kernel, dM/M, or the marginal utility growth, under a regime-switching shock $m(dt, du) - \gamma_m(dt, du)$ given $X(t)$ and $S(t-)$. We refer to this second component as the regime-switching risk premium. This risk premium is present not only because regime shifts have a direct impact on the bond price, $\frac{\Delta_S f}{f}$, but also because regime shifts have a direct impact, $-\lambda_S$, on the pricing kernel or investors' marginal utility. As in the case of the diffusion risk premium, if the regime-switching shocks generate movements in the bond return and the pricing kernel (or marginal utility) in the same direction, the covariance is positive and the risk premium is negative as the bond offers investors a hedge against the risk of regime shifts. On the other hand, If regime shifts generate movements in the bond return and the pricing kernel (or

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[6] It is possible that $\frac{1}{f}\frac{\partial f}{\partial X'}$ depends on $X(t)$ and $S(t)$ as well. Nonetheless in the broad class of affine models, $\frac{1}{f}\frac{\partial f}{\partial X'}$ is a constant that depends only on the bond's maturity.

marginal utility) in the opposite directions, the covariance is negative and the risk premium will be positive. In this case, regime shifts make the bond risky because they decrease the asset's return when investors' marginal utility is high.

In some regime-switching models such as Bansal and Zhou [5], however, it is assumed that regime-switching risk is not priced by fixing λ_S at zero. The assumption is equivalent to assume that regime-switching is not an aggregate risk, and therefore the regime-switching shock $m(dt,du) - \gamma_m(dt,du)$ doesn't have a any impact on $\frac{dM}{M}$ given X and S (see Equation (6)). Since most empirical regime-switching models are motivated by business cycle fluctuations or shifts in monetary policies, it seems important to treat regime shifts as an aggregate risk. Some empirical results such as those from Dai et al. [16], suggest that λ_S not only is statistically significant, but also economically important.

Finally, as we can see from Equation (14), the regime-switching risk premium is in general time-varying. This is simply because both the market price of regime-switching risk λ_S and the regime-switching intensity h can depend on state variables $X(t)$ and $S(t)$. Moreover, as we will show below, the term $\frac{\Delta_S f}{f}$ is also time-varying even in affine models, unlike the constant term $\frac{1}{f}\frac{\partial f}{\partial X}$ in the diffusion risk premium. This property of regime-switching risk premium adds another flexibility to models with multiple regimes.

2.5 An affine regime-switching model

To further illustrate the effects of regime shifts on the term structure of interest rates, we resort to the tractable specifications of affine models that have been widely used in the empirical studies. Duffie and Kan [18] and Dai and Singleton [13] have detailed discussions of affine term structure models under diffusions. Duffie, Pan and Singleton (2000) deals with general asset pricing under affine jump-diffusions. Extensions of the standard affine models are discussed, for example, in Duffee [17] and Duarte (2004) which propose a class of essentially affine or semi-affine models. In models with regime shifts, Landen [37], Bansal and Zhou [5], Evans [23], Dai, Singleton and Yang (2006) and Ang and Bekaert (2007)among others all have similar affine structures. The main advantage of affine models is that they can produce analytical solutions of the term structure of interest rates, and yet at the same time are flexible enough to accommodate time-varying risk premiums and stochastic volatilities. The model we discuss below generalizes the single-regime affine models to include regime-dependent mean reversion coefficients, priced regime-switching risk as well as time-varying regime-switching probabilities.

More specifically, we make the following parametric assumptions:

(1) $\Theta(X(t),S(t-)) = \Theta_0(S(t-)) + \Theta_1(S(t-))X(t)$ where $\Theta_0(S)$ is a $L \times 1$ vector and $\Theta_1(S)$ is $L \times L$ matrix.

(2) $\Sigma(X(t),S(t-))$ is a $L \times L$ diagonal matrix with the ith diagonal element given by $[\Sigma(X_t,S_{t-})]_{ii} = \sqrt{\sigma_{0,i}(S_{t-}) + \sigma'_{1,i}X_t}$ for $i = 1,\cdots,L$ where $\sigma_{1,i}$ is a $L \times 1$ vector. We assume both $\sigma_{0,i}$ and $\sigma_{1,i}$ are positive.

(3) $h(u,X(t),S(t-)) = e^{h_0(u,S_{t-}) + h'_1(u,S_{t-})X_t}$ where $h_1(u,S_{t-})$ is a $L \times 1$ vector.

(4) $\lambda_D(X(t),S(t-)) = \Sigma(X_t,S_{t-})^{-1}\left(\lambda_{0,D}(S_{t-}) + \lambda_{1,D}X_t + \Theta_1(S_{t-})X_t\right)$ where $\lambda_{0,D}(S_{t-})$ is a $L \times 1$ vector, and $\lambda_{1,D}$ is a $L \times L$ matrix that is constant across regimes.

(5) $1 - \lambda_S(u,S(t-),X(t)) = e^{\phi(u,S_{t-})}/h(u,S_{t-},X_t)$ where $h(u,S_{t-},X_t) \neq 0$.

(6) The instantaneous short-term interest $r(t-)$ is a linear function of the state variables $X(t)$ given $S(t-)$

$$r(t-) = \psi_0(S(t-)) + \psi'_1 X(t) \tag{15}$$

where $\psi_0(S)$ is a regime-dependent constant and ψ_1 is a $L \times 1$ vector of constants that are independent of regimes.[7]

The first three assumptions are about the dynamics of the state variables. For $X(t)$, Assumption (1) and (2) implies that its drift and volatility terms are all affine functions of X_t conditional on regimes. In particular $X(t)$ is given by

$$dX(t) = \left[\Theta_0(S(t-)) + \Theta_1(S(t-))X(t)\right]dt + \Sigma(X(t),S(t-))dW(t), \tag{16}$$

where

$$\Sigma(X(t),S(t-)) = \begin{pmatrix} \sqrt{\sigma_{0,1}(S_{t-}) + \sigma'_{1,1}X_t} & & \\ & \ddots & \\ & & \sqrt{\sigma_{0,L}(S_{t-}) + \sigma'_{1,L}X_t} \end{pmatrix}$$

Under this specification, the mean-reversion coefficient and the "steady-state" value of $X(t)$ are given by $-\Theta_1(S)$ and $-\Theta_1(S)^{-1}\Theta_0(S)$, both can shift across regimes. Moreover, the model also has a flexible specification for the volatility of $X(t)$ with regime specific $\sigma_{0,i}(S)$ and stochastic volatility in each regime $\sigma'_{1,i}X(t)$. Some empirical studies have shown that inflation in the U.S. has become less volatile and less persistent in recent years compared to earlier periods, probably due to a combination of the moderation of output volatility and changes in the monetary policy that has provided a better anchor for long-run inflation expectations. If the latent factors in $X(t)$ are to capture the fundamental driving forces in the economy, it is important to allow this kind of regime shifts in an empirical model of the term structure of interest rates.[8]

Assumption (3) implies that the log intensity of regime shifts is an affine function of the state variable X_t conditional on regimes. This assumption ensures the

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[7] If ψ_1 is regime-dependent, an analytical solution of the yield curve is in general unavailable. Bansal and Zhou [5] and Wu and Zeng (2006) assume that ψ_1 depends on regimes and obtain the term structure of interest rates under log-linear approximation.

[8] In order to obtain a closed form solution of the term structure of interest rates, we need to restrict σ_1 to be constant across regimes.

positivity of the intensity function and also allows the transition probability to be time-varying.

The next two assumptions deal with the market prices of risk. In the standard affine models, the market price of (diffusion) risk is assumed to be proportional to the volatility of the state variable X_t. Such a structure is intuitive: risk compensation goes to zero as risk goes to zero. However, since variances are nonnegative, this specification limits the variation of the compensations that investors anticipate to receive when encountering a risk. More precisely, since the compensation is bounded below by zero, it cannot change sign over time. This restriction, however, is relaxed in the essentially affine models of Duffee [17]. Dai and Singleton [14] argues that this extension is crucial for empirical models to account for the failure of the expectation theory of the term structure of interest rates.

Following this literature, we use a similar specification as that of essentially affine models for the market price of the diffusion risk in Assumption (4), but with an extension to include multiple regimes. Under this assumption, conditional on regimes, the diffusion risk premium of bonds will be proportional to $\lambda_{0,D}(S(t-)) + \lambda_{1,D}X(t) + \Theta_1(S(t-))X(t)$, a linear function of the state variable $X(t)$. Moreover Assumption (4) implies that, from Lemma 2, $\lambda_{1,D}$ is the *risk neutral* mean reversion coefficient of $X(t)$, which is assumed to be constant across regimes. It turns out this is one of the crucial conditions that are necessary for obtaining closed form solutions of the term structure of interest rates under regime shifts.[9]

For the market price of regime switching risk λ_S, Assumption (5) postulates that, conditional on regimes, $1 - \lambda_S$ is proportional to the inverse of regime-switching intensity. Under this assumption, λ_S can be time-varying, and the higher the regime-switching intensity, the higher the risk compensation. We restrict λ_S to take this particular form because it implies that the *risk neutral* regime-switching intensity is constant conditional on regimes, $\tilde{h}(u, S_{t-}, X_t) = e^{\phi(u,S_{t-})}$. This is another restriction we need to impose on the model in order to obtain a closed-form solution of the term structure of interest rates.

Proposition 2. *Under the assumption (1)-(6), the price at time $t-$ of a default-free pure discount bond with maturity τ is given by $P(t-, \tau) = e^{A(\tau,S_{t-})+B(\tau)'X_t}$ and the τ-period interest rate is given by $R(t-, \tau) = -\frac{A(\tau,S_{t-})}{\tau} - \frac{B(\tau)'X_t}{\tau}$, where $B(\tau) = (B_1(\tau), \cdots, B_L(\tau))'$, and $A(\tau,S)$ and $B(\tau,S)$ are given by the following ordinary integral-differential equations*

$$-\frac{\partial B(\tau)}{\partial \tau} - \lambda_{1,D}'B(\tau) + \frac{1}{2}\Sigma_1'B^2(\tau) = \psi_1 \tag{17}$$

and

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[9] In the regime switching model of Bansal and Zhou [5], the risk-neutral mean reversion coefficient is allowed to shift across regimes. But the term structure of interest rates can only be solved analytically under log linear approximation.

$$-\frac{\partial A(\tau,S)}{\partial \tau} + B(\tau)'[\Theta_0(S) - \lambda_{0,D}(S)] + \frac{1}{2}B(\tau)'\Sigma_0(S)B(\tau)$$
$$+ \int_U \left[e^{A(\tau,u)-A(\tau,S)} - 1 \right] e^{\phi(u,S)} \eta(du) = \psi_0(S) \tag{18}$$

with boundary conditions $A(0,S) = 0$ and $B(0) = 0$, where $B^2(\tau) = (B_1^2(\tau), \cdots, B_L^2(\tau))'$, and Σ_1 and $\Sigma_0(S)$ are $L \times L$ matrices given by

$$\Sigma_1 = \begin{pmatrix} \sigma'_{1,1} \\ \vdots \\ \sigma'_{1,L} \end{pmatrix}, \ and \ \Sigma_0(S) = \begin{pmatrix} \sigma_{0,1} & & \\ & \ddots & \\ & & \sigma_{0,L} \end{pmatrix}. \tag{19}$$

2.6 The effects of regime shifts on the yield curve

With the analytical solution in Proposition 2, we can now further illustrate the effects of regime shifts on the term structure of interest rates. First, the general result for bond risk premiums in Equation (14) is now simplified as

$$E_t \left(\frac{dP_t}{P_{t-}} \right) - r_t dt = [\lambda'_{0,D}(S_{t-}) + X_t' \lambda'_{1,D} + X_t' \Theta_1'(S_{t-})]B(\tau)dt$$
$$+ \int_U \left(e^{A(\tau,u)-A(\tau,S_{t-})} - 1 \right) (e^{h_0(u;S_{t-})+h_1'(u;S_{t-})X_t} - e^{\phi(u;S_{t-})})\eta(du)dt. \tag{20}$$

As in Equation (14), the first term on the right hand side of equation (20) is the diffusion risk premium and the second term is the regime-switching risk premium. In the standard affine models without regime shifts, the risk premium is determined by a linear function of the state variable $X(t)$, that is $[\lambda'_{0,D} + X_t' \lambda'_{1,D} + X_t' \Theta_1']B(\tau)$.[10] Moreover, only the factor loadings in the term structure of interest rates, $B(\tau)$, affect the risk premium. The intercept term, $A(\tau)$, doesn't enter the above equation.

By introducing regime shifts, Bansal and Zhou [5] B essentially makes the risk premium a non-linear function of the state variable $X(t)$ because the intercept term $\lambda_{0,D}$ and the slope coefficient $\lambda_{1,D} + \Theta_1$ are now both regime-dependent. Bansal and Zhou [5] shows that it is mainly this feature of their model that provides improved goodness-of-fit over the existing term structure models. One restriction of Bansal and Zhou [5], however, is that they assume that the regime-switching risk is not priced, $\lambda_S(u;S,X) = 0$. In the context of the above affine model, this is equivalent to assume that $e^{h_0(u;S)+h_1'(u;S)X} = e^{\phi(u;S)}$, that is the risk neutral regime-switching probabilities, $e^{\phi(u;S)}$, is the same as the physical regime-switching probabilities $e^{h_0(u;S)+h_1'(u;S)X}$. Therefore the bond risk premium is still a linear function of the state variable *conditional on regimes*.

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[10] In the more restrictive CIR models, $\lambda'_{0,D}(S_{t-}) + X_t' \lambda_{1,D} + X_t' \Theta_1(S)]$ is further restricted to be proportional to the variance of the state variables.

In Dai et al. [16] and Ang and Bekeart (2007), $\lambda_S(u, S, X)$ is not restricted to be zero.[11] Equation (20) shows that this extension provides an additional flexibility for the model to account for time-varying risk premiums observed in the data, because the second-term on the right-hand side of Equation (20) is a highly non-linear function of the state variable $X(t)$ even conditional on regimes. The equation also shows that the regime-switching risk, λ_S, directly affects the term structure of interest rates through the intercept term, $A(\tau, S)$,[12] while the diffusion risk, λ_D, affect the yield curve through the factor loading, $B(\tau)$.[13]

One caveat of the affine models such as the one obtained in Proposition 2, however, is the tension between the transition probabilities between regimes and the market price of regime-switching risk. To obtain a closed form solution, affine models have to restrict the transition probabilities across regimes under the *risk neutral probability measure* to be constant. In other words, $\tilde{h}(u, S(t-), X(t))$ needs to be independent of $X(t)$. In Equation (20), that term is given by $e^{\phi(u, S_{t-})}$. On the other hand, the transition probabilities under the *the physical measure*, $h(u, S(t-), X(t))$, are given by $e^{h_0(u; S_{t-}) + h_1'(u; S_{t-})X_t}$ in Equation (20). If the model allows the transition probabilities under the physical measure to be time varying, i.e. $h_1 \neq 0$, as many regime-switching models do,[14] we would impose that the regime-switching risk is priced. This is because $\lambda_S(u, S, X)h(u, S, X) = h(u, S, X) - \tilde{h}(u, X, S)$, and in affine models $\lambda_S(u, S, X)h(u, S, X) = e^{h_0(u; S_{t-}) + h_1'(u; S_{t-})X_t} - e^{\phi(u, S_{t-})}$, which implies that $\lambda_S(u, S, X)$ is not zero and must be time-varying as well. It is possible to loose this restriction with more general models (i.e. with time-varying regime-switching probabilities and zero market price of regime-switching risk), but probably at the cost of not being able to obtain a closed form solution to the term structure of interest rates.

One stylized fact of the term structure of interest rates is that long-term interest rates do not attenuate in volatility. In affine models without regime shifts, interest rates are given by $R(t, \tau) = -\frac{A(\tau)}{\tau} - \frac{B(\tau)'X_t}{\tau}$. Therefore the volatility of interest rates is determined by the factor loading $-\frac{B(\tau)}{\tau}$ alone, where $B(\tau)$ is given by the differential equation (17). To illustrate why the volatility of long-term interest rates might pose a challenge to affine models, let's consider the one-factor Gaussian model for example. In this case, the solution to $B(\tau)$ depends on the value of $\lambda_{1,D}$. If $\lambda_{1,D} \gg 0$, $B(\tau)$ will converge very quickly (at the rate of $e^{-\tau\lambda_{1,D}}$) to a constant as τ increases, hence so does the interest rate $R(t, \tau)$. In order to generate volatile long-term interest rates, we need $\lambda_{1,D} \approx 0$, which implies $B(\tau) \approx -\tau$ and $R(t, \tau) \approx -\frac{A(\tau)}{\tau} + X(t)$. Long-term interest rates would be as volatile as short-term interest rates. But note

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[11] In Ang and Bekeart (2007), however, the market price of regime-switching risk is not explicitly defined. $\lambda_S(u, S, X)$ can be derived from the specification of the pricing kernel.

[12] Of course, $A(\tau, S)$ also depends on the factor loading $B(\tau)$ through the differential equation (18)

[13] See Siu [44] for a discussion of the pricing of regime-switching risk in equity market.

[14] Hamilton [34] and Filardo [25] are examples of regime-switching models of business cycles with time-varying transitions probabilities. In regime-switching models of interest rates, time-varying transition probabilities are assumed in Gray [32], Boudoukh et al. [9] and more recently Dai et al. [16] among others.

that $\lambda_{1,D}$ is the mean reversion coefficient of the state variable $X(t)$ under the risk-neutral probability measure. The requirement that $\lambda_{1,D} \approx 0$, therefore, is to assume that $X(t)$ is close to a unit root process under the risk-neutral probability measure.

In models with regime shifts, $-\frac{A(\tau,S(t-))}{\tau}$ is stochastic and adds another source of volatility for the interest rate $R(t,\tau)$. Since the volatility of $-\frac{A(\tau,S(t-))}{\tau}$ will not attenuate, this would translate directly into the volatility of long-term interest rates even if $\lambda_{1,D} \gg 0$.

3 Empirical results

3.1 Data and summary statistics

The data used in this study are monthly interest rates from June 1964 to December 2001 obtained from the Center for Research in Security Prices (CRSP).[15] These are yields on zero-coupon bonds extracted from U.S. Treasury securities. There are eight interest rates with maturities ranging from 1 month to 5 years. Table 1 contains their summary statistics. We can see that the yield curve is on average upward-sloping and the large skewness and kurtosis suggest significant departure from Gaussian distribution. As Timmermann [46] shows, Markov switching models can generate such large skewness and kurtosis. Another feature of the data is that long-term interest rates (for example, the 5-year rate) are almost as volatile as the 1-month rate. Moreover, volatilities of the interest rates have a hump-shaped structure as noted in Dai et al. [16]. The standard deviation increases from 2.45% for the 1-month rate to 2.60% for the 6-month rate, and then declines to 2.32% for the 5-year rate. Also note that all interest rates are very persistent with high auto-correlation coefficients. The 6-month and 5-year rate are plotted in Figure 1.

Table 1 Interest rates summary statistics

	1M	3M	6M	1Y	2Y	3Y	4Y	5Y [a]
Mean	0.0594	0.0638	0.0659	0.0681	0.0702	0.0717	0.0729	0.0735
Std. Dev.	0.0245	0.0258	0.0260	0.0252	0.0246	0.0238	0.0235	0.0232
Maximum	0.1614	0.1603	0.1652	0.1581	0.1564	0.1556	0.1582	0.1500
Minimum	0.0164	0.0171	0.0178	0.0192	0.0237	0.0296	0.0336	0.0367
Skewness	1.4278	1.3717	1.3122	1.1737	1.1288	1.1283	1.1003	1.0565
Kurtosis	5.4659	5.1336	4.9150	4.4157	4.1226	4.0313	3.9196	3.7344
Auto Corr	0.947	0.971	0.971	0.970	0.976	0.978	0.979	0.981

[a] 1M, 3M, 6M indicate 1-month, 3-month and 6-month interest rates respectively. 1Y, 2Y, ..., 5Y indicate 1-year, 2-year, ... and 5-year interest rate respectively.

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[15] To make our study comparable, we consider the roughly same sample period as that in Bansal and Zhou [5].

We report in Table 2 the results from the standard regressions regarding the expectation hypothesis, which states that a long-term interest rate is just the average of the expected short-term interest rate over the life of the long-term bond. The regression used to test this hypothesis is

$$i(R_{t+j}^i - R_t^{i+j}) = \alpha + \beta_{ij}[j(R_t^{i+j} - R_t^j)] + \varepsilon_{t+j}^i$$

where R_t^k is the k-period interest rate at time t. Under the null hypothesis of the expectation theory, $\beta_{ij} = 1$. However, as it is well known, most regressions produce estimates of β_{ij} that are significantly less than 1, and often are negative. Table 2 confirms this stylized fact. The estimates of β_{ij} are either insignificantly different from 0 or significantly negative. It is interesting to note, however, the yield spread between 5-year and 4-year rate predicts the future (4 years ahead) 1-year rate with correct sign and the expectation hypothesis can not be rejected. In fact, as maturity increases, the estimate of β_{ij} tends to increase from negative to positive, suggesting that the expectation hypothesis might hold in longer terms.

Table 3 contains correlation coefficients between the expected excess bond returns and a business cycle dummy variable, *BC*. NBER dates of business cycles
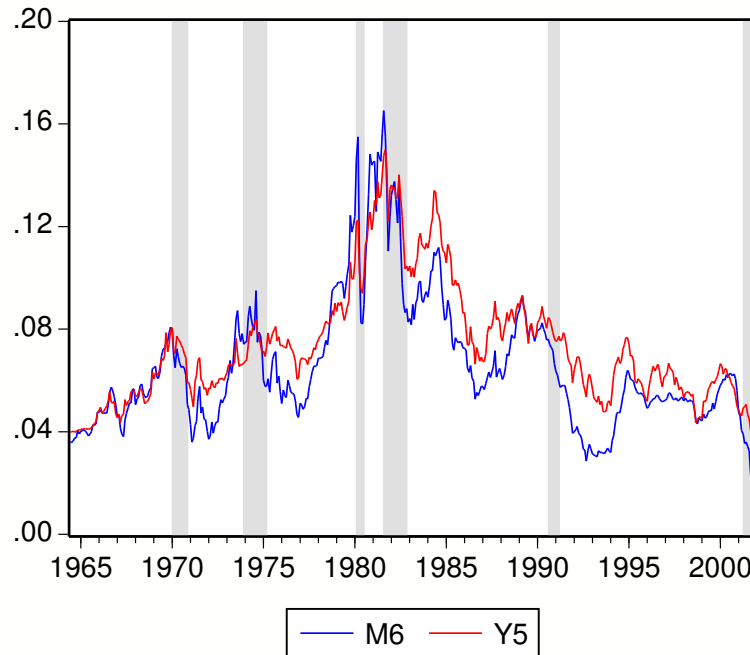

Fig. 1 Historical interest rates and the business cycle. The figure plots the 6-month (series M6) and 5-year (series Y5) interest rates during 1964 - 2001. NBER business cycle recessions are indicated the shaded area.

Table 2 Expectation-hypothesis regression [a]

	i+j=0.5	i+j=1	i+j=2	i+j=3	i+j=4	i+j=5
j=0.25	-0.8656					
	(0.3766)					
j=0.5		-0.6564				
		(0.4669)				
j=1			-0.8769	-1.2513	-1.6585	-1.6085
			(0.3932)	(0.4697)	(0.5270)	(0.6084)
j=2				-0.3395	-0.8568	-1.0011
				(0.5879)	(0.6337)	(0.7539)
j=3					0.0554	-0.0619
					(0.4221)	(0.5583)
j=4						0.6853
						(0.4681)

[a] This table reports the estimate of β_{ij} in regression $i(R^i_{t+j} - R^{i+j}_t) = \alpha + \beta_{ij}[j(R^{i+j}_t - R^j_t)] + \varepsilon^i_{t+j}$ where R^k_t is the k-year interest rate at time t. Under the null hypothesis of the expectation theory, $\beta_{ij} = 1$. Numbers in parentheses are Newy-White standard errors.

are used to distinguish between expansions ($BC = 1$) and recessions ($BC = 0$). Excess bond returns are obtained as the differences between holding-period returns (1-month) on long-term bonds (1, 2, ..., 5-year bonds respectively) and the 1-month interest rate.[16] We can see from Table 3 that the correlation coefficients are all negative, which is consistent with the counter-cyclical behavior of risk premiums as documented in Fama and French [24]. Alternatively we can regress the ex-post excess bond returns on the business cycle dummy and the yield spread of the previous period. We include the yield spread in the regression because empirical studies have suggested that yield spreads or forward rates contain information about the state variables that drive the interest rates. Again, the regression coefficients on the business cycle dummy variable are all negative and significant, confirming the counter-cyclical property of bond risk premiums. Interestingly, if we don't include the business cycle dummy variable in the regressions, estimates of the coefficient on the yield spread are all positive and significant (not reported in Table 3), indicating that yield spreads do forecast bond returns. Once we include the business cycle dummy in the regressions, however, the dummy variable completely drives out the predicating power of yield spreads for bond returns.

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[16] Continuously compounded bond returns are $H_{t+\Delta t} \equiv -(\tau - \Delta t)R_{t+\Delta t}(\tau - \Delta t) + \tau R_t(\tau)$, where $R_t(\tau)$ is the yield on a τ-year bond at time t. Since we don't have data on $R_{t+\Delta t}(\tau - \Delta t)$, we approximate it by $R_{t+\Delta t}(\tau)$ for $\tau \gg \Delta t$, where $\Delta t = 1$ month. Also note that $\text{Corr}[E_t(H_{t+\Delta t}), BC_t] = \text{Corr}(H_{t+\Delta t}, BC_t)$ under rational expectations.

Table 3 Correlation among bond returns and the business cycle [a]

	RETY1	RETY2	RETY3	RETY4	RETY5	BC(-1)
RETY1	1					
RETY2	0.9423	1				
RETY3	0.9057	0.9632	1			
RETY4	0.8515	0.9328	0.9592	1		
RETY5	0.8506	0.9289	0.9553	0.9655	1	
BC(-1)	-0.1986	-0.1703	-0.1518	-0.1277	-0.1218	1
$\hat{\alpha}_i$	0.8966	1.4411	0.0419	0.3453	0.3283	
	(0.4320)	(1.2241)	(1.5286)	(1.8086)	(1.8807)	
$\hat{\beta}_i$	−0.0335**	−0.0596**	−0.1153**	−0.1202**	−0.1270**	
	(0.0096)	(0.0213)	(0.0298)	(0.0383)	(0.0443)	
$\hat{\gamma}_i$	-1.0916	0.1590	2.4134	2.7410	2.8611	
	(0.9559)	(1.3883)	(1.7150)	(2.0863)	(2.3112)	
Adjusted R^2	0.048	0.041	0.042	0.036	0.033	

[a] The first 6 rows of this table report the sample correlation coefficients among the excess bond returns and the business cycle. RETY1, RETY2, ..., RETY5 are the *ex-post* holding-period (1-month) returns on 1-year, 2-year, ..., 5-year bonds minus the 1-month interest rate, respectively. BC is the dummy variable for the business cycle with BC=1 indicating an expansion and BC=0 indicating a recession. The last 4 rows report the OLS regression $\text{RETY}i_t = c + \alpha_i \text{SP}_{i,t-1} + \beta_i \text{BC}_{t-1} + \gamma_i[\text{SP}_{i,t-1} \times \text{BC}_{t-1}] + \varepsilon_{i,t}$, where $\text{RETY}i$ is the holding-period excess return on a i-year bond, SP_i is the yield spread between the i-year bond and the 1-month Bill. Numbers in parentheses are Newy-West standard errors. An $**$ indicates the estimate is significant at 5% level.

3.2 Estimation procedure

The econometric methodology we adopt to estimate the term structure model in Proposition 2 is Efficient Method of Moments (EMM) proposed in Bansal et al (1995) and Gallant and Tauchen [28, 30].[17] We assume that there are two distinct regimes ($N = 2$) for $S(t)$. Therefore (17) and (18) define a system of 3 differential equations that must be solved simultaneously. Under regime shifts, the number of parameters of the model increases quickly with each additional factor. In this paper, we estimate a two-factor version of the model, and fit the model to the 6-month and the 5-year interest rates as in Bansal and Zhou [5].

Under EMM procedure, the empirical conditional density of the observed interest rates is first estimated by an auxiliary model that is a close approximation to the true data generating process. Gallant and Tauchen [30] suggests a semi-nonparametric (SNP) series expansion as a convenient general purpose auxiliary model. As pointed out by Bansal and Zhou [5], one advantage of using the semi-nonparametric specification for the auxiliary model is that it can asymptotically converge to any smooth distributions (see also Gallant and Tauchen [29]), including the density of Markov regime-switching models. The dimension of this auxiliary model can be selected

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[17] Bansal and Zhou [5], Bansal et al. [6] are excellent examples of applying EMM to estimate the term structure model under regime shifts. Dai and Singleton [13] also provides extensive discussions of estimating affine term structure models using EMM procedure.

by, for example, the Schwarz's Bayesian Information Criterion (BIC)[18]. Table 4 reports the estimation results of the preferred auxiliary semi-nonparametric model. The score function of the auxiliary model are then used to generate moment conditions for computing a chi-square criterion function, which can be evaluated through simulations given the term structure model under consideration. A nonlinear optimizer is used to find the parameter setting that minimizes the criterion function. Gallant and Tauchen [28] shows that such estimation procedure yields fully efficient estimators if the score function of the auxiliary model encompasses the score functions of the model under consideration.

Without further normalization, however, Dai and Singleton [13] shows that affine models as the one in Proposition 2 are under-identified. Therefore we restrict in EMM estimation that $\Theta_1(S)$ and $\lambda_{1,D}$ are lower triangular matrixes. We also restrict $\psi_1 = (1,0)'$ and fix $\psi_0(S) = 0$ so that the instantaneous short-term interest rate $r_t = x_{1,t}$. In other words we restrict that the first state variable is simply the short-term interest rate. Because of the relation between the regime-switching risk and the transition probabilities as discussed in Section 2.6, we assume that the transition probabilities are not time-varying, therefore we don't force the regime-switching risk to be priced. In other words, we fix $h_1(u,S) = 0$. Under these restrictions, the model has 29 parameters. After initial estimation, we further fix at zero those parameters whose estimates are close to 0 and statistically insignificant, and re-estimate the model. The final results are reported in Table 5.

3.3 Discussions

Many empirical models of interest rates that incorporate regime switching are motivated by the recurrent shifts between different phases of the business cycle experienced by the aggregate economy. It is therefore interesting to see how the latent regimes in our model of the term structure of interest rates correspond to the business cycle. Following the approach in Bansal and Zhou [5], we first compute interest rates of different maturities conditional on each regime, $\hat{R}(t,\tau|S_t)$, using the estimated term structure model reported in Table 5. An estimate of S_t is then obtained by choosing the regime that minimizes the differences between the actually observed interest rates $R(t,\tau)$ and $\hat{R}(t,\tau|S_t)$ or the pricing errors, that is, $\hat{S}_t = \arg\min \sum_\tau |R(t,\tau) - \hat{R}(t,\tau|S_t)|$. The estimated regimes are plotted in Figure 2 together with the business cycle expansions and recessions identified by NBER. Consistent with the result in Bansal and Zhou [5], the figure clearly shows that the regimes underlying the dynamics of the term structure of interest rates are intimately related to the fluctuations of the aggregate economy. Our model is able to identify all six recessions in the sample period. The result is also consistent with the findings from some earlier empirical studies, such as Estrella and Mishkin [22] and Chau-

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[18] As for model selection for regime switching models (or general Hidden Markov models), Scott [42] gave an excellent review on limitations of various criteria, including BIC and AIC.

Table 4 Parameter estimates of SNP density [a]

Parameter	Estimates	Standard Error
a(0,0)	1.00000	0.00000
a(0,1)	0.30987	0.07457
a(1,0)	0.05612	0.04619
a(0,2)	-0.35596	0.06619
a(1,1)	0.10687	0.02843
a(2,0)	-0.04299	0.03182
a(0,3)	-0.02352	0.01820
a(3,0)	-0.01374	0.01023
a(0,4)	0.03257	0.01022
a(4,0)	0.02216	0.00514
$\mu(1,0)$	-0.07309	0.01232
$\mu(2,0)$	-0.05375	0.00800
$\mu(1,1)$	0.81742	0.03382
$\mu(1,2)$	0.15274	0.03288
$\mu(2,1)$	0.00221	0.03763
$\mu(2,2)$	0.95765	0.03690
R(1,0)	0.01843	0.00295
R(2,0)	0.15853	0.01250
R(3,0)	0.18509	0.01616
R(1,1)	0.17234	0.02936
R(2,1)	0.11702	0.05355
R(1,2)	0.12163	0.02951
R(2,2)	0.09551	0.05472
R(1,3)	0.02379	0.03465
R(2,3)	0.10629	0.05286
R(1,4)	0.04649	0.02601
R(2,4)	0.07097	0.04875
R(1,5)	0.05068	0.02554
R(2,5)	0.02903	0.04716

[a] This table reports point estimates as well as their standard errors of the parameters in the preferred SNP model according to BIC (BIC=-1.2488, AIC=-1.3786). $a(i,j)$ are parameters of the Hermit polynomial function. $\mu(i,j)$ are parameters of the VAR conditional mean. $R(i,j)$ are parameters of the ARCH standard deviation of the innovation z. See Gallant and Tauchen [30] or Bansal and Zhou [5] for more detailed interpretations of these parameters.

vet and Potter [11] among others, that the yield curve has a significant predicative power for the turning point of the business cycle.

The point estimates reported in Table 5 confirm that regime-switching indeed seems to be an important feature of interest rate dynamics. For example, according to the estimated parameters, the first factor has a lower long-run level (2.8%) and smaller mean-reverting coefficient (0.1743), hence higher persistence, in Regime 1 than in Regime 2 where the long-run level is 12.89% with a mean-reverting coefficient 0.2150. Since the volatility of the factor is specified to be proportional to its level, these estimates imply that the factor exhibits higher volatility in Regime 2 than in Regime 1. The estimates of $\sigma_{0,2}$ in Regime 1 and Regime 2 suggest that the

second factor also has higher volatility in Regime 2 than in Regime 1. These results are consistent with early findings (Ang and Bekaert [3], for example) with regard to the persistence and volatility of interest rates across different regimes. We plot in Figure 3 and 4 the estimated mean yield curve together with the observed average yield curve in both regimes. We also plot in Figure 5 the standard deviations of the estimated interest rates together with the corresponding sample standard deviations in both regimes. We can see that our model fits the yield curve data reasonably well. The yield curve is upward sloping on average in Regime 1 and is flat or slightly downward sloping in Regime 2. Moreover, interest rates are less volatile in Regime 1 than in Regime 2. One stylized fact of the yield curve is that interest rate volatility doesn't seem to attenuate as maturity increases. Figure 5 shows that this non-attenuating volatility is mainly a Regime 1 phenomenon where the volatility is relatively low. On the other hand, in the high-volatility regime (Regime 2), interest rate volatility does decline significantly as maturity increases.

In Table 5, $\Theta_0^Q(S)$ and $\Theta_1^Q(S)$ are the risk-neutral counterparts of $\Theta_0(S)$ and $\Theta_1(S)$. They imply that the coefficients of the market price of diffusion risk are $\lambda_{0,D}(S) = \Theta_0(S) - \Theta_0^Q(S)$ and $\lambda_{1,D} = -\Theta_1^Q$. Note that in order to obtain a closed-form solu-
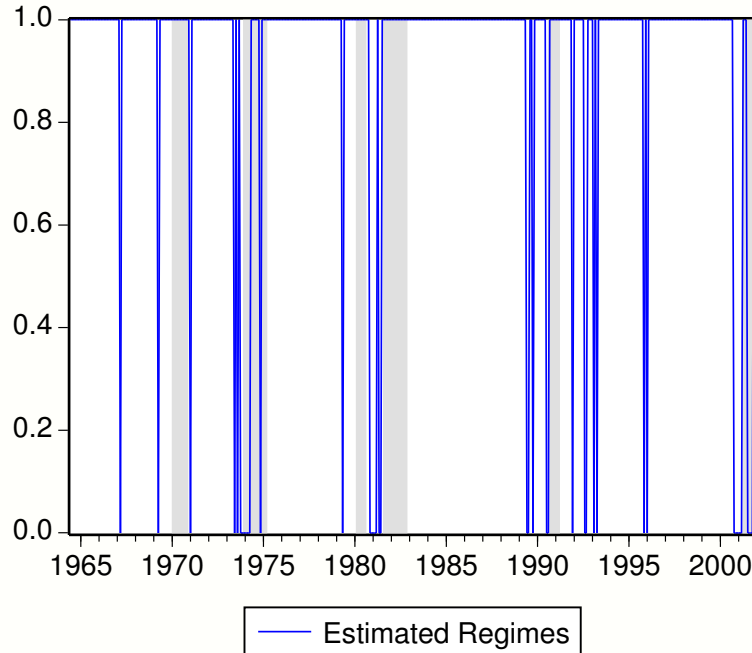

Fig. 2 Estimated term structure regimes during 1964-2001. The figure plots the estimated interest rates during 1964-2001. The two regimes are coded as 1 and 0. The shaded areas are economic recessions dated by NBER.

tion of the terms structure of interest rates, we have restricted Θ_1^Q, hence $\lambda_{1,D}$, to be independent of regimes. Nonetheless, our model still allows $\lambda_{0,D}$, hence the market price of risk, to change across regimes, and the estimates of $\Theta_0^Q(S)$ and $\Theta_0(S)$ confirm that the market price of risk in Regime 1 is indeed very different from that in Regime 2. $\lambda_{0,D} = 0.0018$ in Regime 1 and $\lambda_{0,D} = 0.0262$ in Regime 2. Bansal and Zhou[5] also finds evidence that the market price of risk is regime-dependent and shows that it is this feature of the market price of risk that accounts for the improved empirical performance of their model over the existing ones. The difference between the model in Bansal and Zhou[5] and the present one is that Bansal and Zhou[5] restricts $\lambda_{0,D}$ to be zero and allows $\lambda_{1,D}$ to be regime-dependent, and as a result, a closed-form solution of the term structure of interest rate can only be obtained using log-linear approximations.

The negative regression coefficients reported in Table 2 strongly reject the expectation hypothesis of the term structure of interest rates. Researchers have often pointed to the presence of time-varying risk premiums as the main cause for this stylized fact about interest rates (see, for example, Dai and Singleton [14]). The model of the term structure of interest rates in the present paper has a very flex-
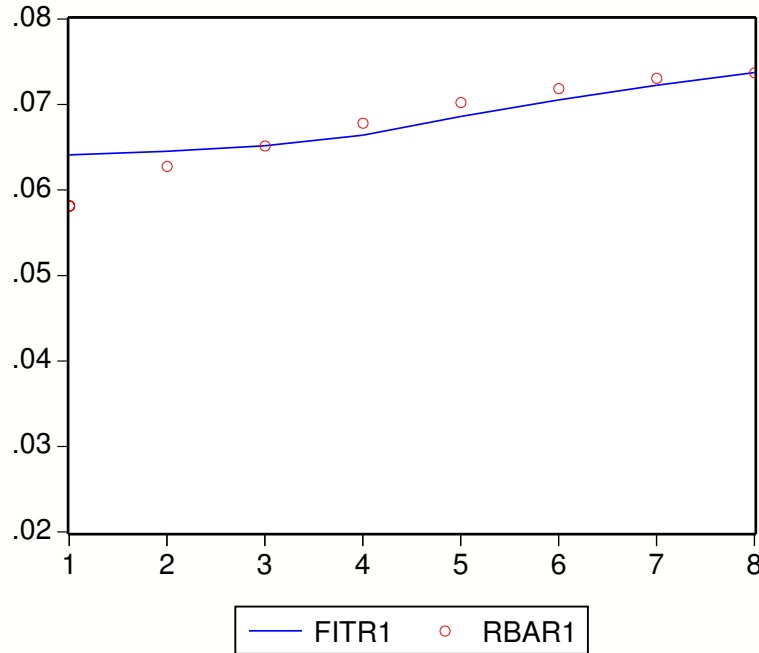

Fig. 3 Estimated yield curve in regime 1. The figure plots the average of the estimated yield curve (FITR1) regime 1. RBAR1 are the average interest rates during 1964-2001 in regime 1. Interest rate maturity ranges from 1 month to 5 years.

ible specification of bond risk premiums. As Equation (20) shows, the risk premium is time-varying first because it is a function of the underlying state variable $X(t)$ as in the standard affine models. Regime-switching, however, makes the coefficients of the risk premium vary across regimes, therefore adds another source of time-variation. Moreover, our model allows for the regime-switching risk to be priced, therefore introduces a new component to bond risk premiums that is also time-varying. To see how this flexible specification of bond risk premiums helps account for the stylized fact of the term structure of interest rates, we use the estimated model to simulate interest rates of various maturities and run the same expectation-hypothesis regressions as those reported in Table 2. The results are included in Table 6. We can see that our model is able to replicate the negative regression coefficients typically found in the literature.

Another stylized fact about the yield curve is that bond risk premiums are typically counter-cyclical as reported in Table 3. For 1-year to 5-year bonds, the risk premiums are all negatively correlated with the business cycle dummy variable. Moreover, simple regressions of the bond risk premiums on the business cycle dummy variable and the yield spread also produce significantly negative coefficients on the
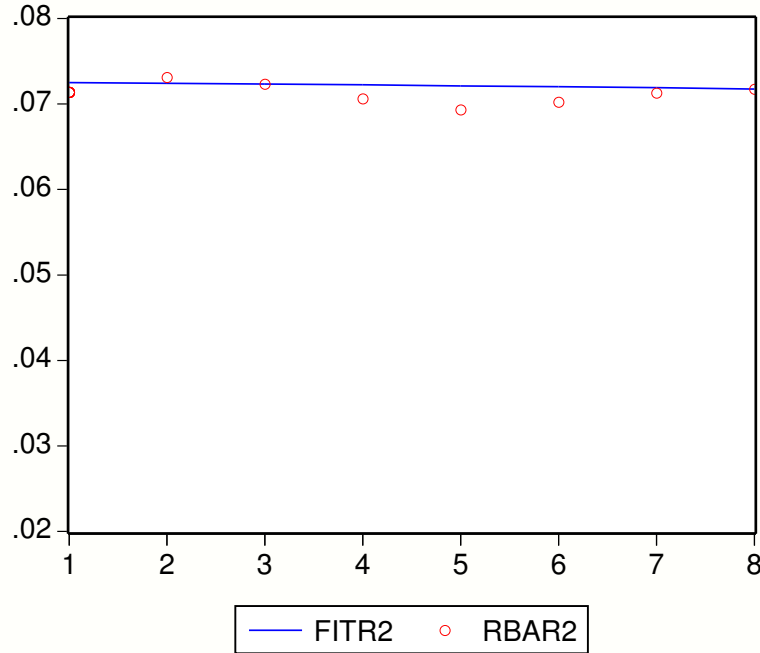

Fig. 4 Estimated yield curve in regime 2. The figure plots the average of the estimated yield curve (FITR2) regime 2. RBAR2 are the average interest rates during 1964-2001 in regime 2. Interest rate maturity ranges from 1 month to 5 years.

business cycle dummy variable in all cases. The dynamic model of term structure of interest rates in the present paper allows us to estimate the instantaneous expected excess return on a long-term bond based on Equation (20). We can then compute the the same correlation coefficients as well as the simple regressions as those in Table 3 using the estimated bond risk premiums. The results are reported in Table 7. We can see that the estimated risk premiums of different bonds are highly and positively correlated as in the data. The correlation coefficients between the estimated risk premium and the business cycle dummy variable, however, are all negative and are similar in magnitude to those found in the data. For example, for the 2-year bond, the correlation coefficient is -0.1703 in the data. Using the estimated risk premiums, the correlation coefficient is -0.1680. We also regress the estimated risk premiums on the business cycle dummy variable and the yield spread. As in the data, the regression coefficients on the business cycle dummy variable are all negative and significant (except for the 1-year bond) in Table 7. A difference, though, is that the yield spread seems to retain its predictive power for bond returns even in the presence of the business cycle dummy variable when the estimated bond risk
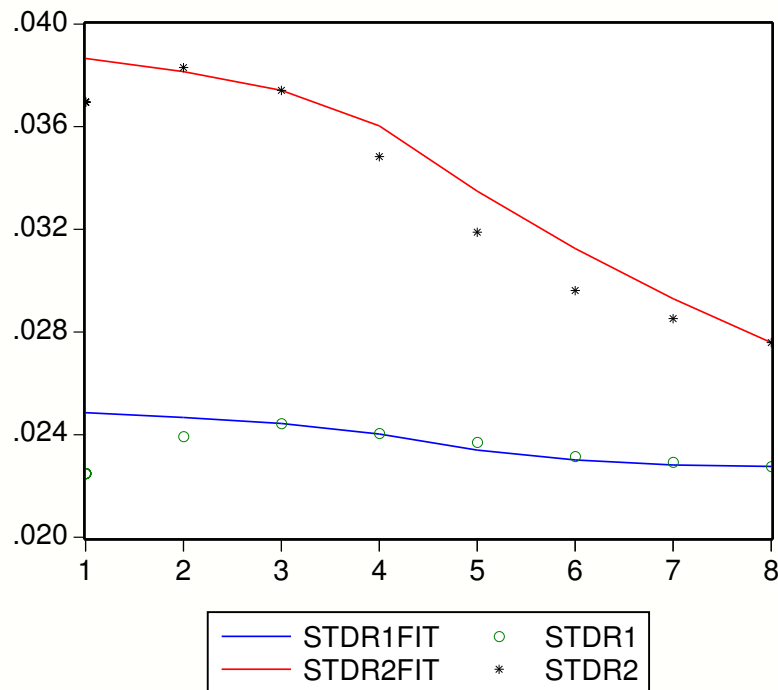

Fig. 5 Interest rate volatility in two regimes. The figure plots the standard deviations of the fitted yield curve (STDR1FIT, STDR2FIT) in regime 1 and 2 respectively. STDR1 and STDR2 are the sample standard deviations of the interest rates in the two regimes. Interest rate maturity ranges from 1 month to 5 years.

Table 5 Parameter estimates of the term structure model [a]

		Regime 1	Regime 2
$\Theta_0(S)$	$\Theta_{0,1}$	0.0050 (0.0021)	0.0277 (0.0025)
	$\Theta_{0,2}$	0.0002 (0.0038)	-0.0003 (0.0014)
$\Theta_1(S)$	$\Theta_{1,11}$	-0.1743 (0.0464)	-0.2150 (0.0049)
	$\Theta_{1,12}$	0	0
	$\Theta_{1,21}$	0.0198 (0.0376)	-0.0887 (0.0065)
	$\Theta_{1,22}$	-0.8476 (0.1480)	-0.6940 (0.1946)
$\Sigma_0(S)$	$\sqrt{\sigma_{0,1}}$	0	0
	$\sqrt{\sigma_{0,2}}$	0.0054 (0.0006)	0.0091 (0.0002)
Σ_1	Σ_1^{11}	0.0018 (0.0001)	0.0018
	Σ_1^{12}	0	0
	Σ_1^{21}	0	0
	Σ_1^{22}	0	0
$\Theta_0^Q(S)$	$\Theta_{0,1}^Q$	0.0032 (0.0002)	0.0015 (0.0009)
	$\Theta_{0,2}^Q$	0.0002	-0.0003
Θ_1^Q	$\Theta_{1,11}^Q$	-0.0184 (0.0057)	-0.0184
	$\Theta_{1,12}^Q$	0	0
	$\Theta_{1,21}^Q$	0.5467 (0.0084)	0.5467
	$\Theta_{1,22}^Q$	-0.1203 (0.0024)	-0.1203
	h_0	-1.6458 (0.0318)	-1.2675 (0.0320)
	ϕ	-0.5403 (5.0976)	-0.3500 (2.3850)
$\chi^2 = 23.42$		zvalue=5.03	d.o.f.=6

[a] This table reports the EMM estimation result of the term structure model in Section 2.5. Numbers in parentheses are standard errors. If an estimate is reported without a standard error, it means that the parameter is not estimated, but fixed at that particular value. Regime-dependent parameters are identified by by their dependence on S. Parameter definitions can be found in Section 2.5. In particular, $\Theta_0^Q(S)$ and Θ_1^Q are the risk-neutral counterparts of $\Theta_0(S)$ and $\Theta_1(S)$ respectively. This simply implies that $\lambda_{0,D} = \Theta_0(S) - \Theta_0^Q(S)$ and $\lambda_{1,D} = -\Theta_1^Q$. Note that Θ_1^Q (hence $\lambda_{1,D}$) is *not* regime-dependent.

premiums are used in the regression (Table 7), whereas the business cycle dummy variable completely drives out the predicating power of yield spreads in the data (Table 3)

One interesting question about the term structure models under regime shifts is that whether or not the regime-switching risk is priced. In our model, the market price of regime-switching risk, $\lambda_S(u,S,X)$, is given by $\lambda_S(u,S,X) = 1 - e^{\phi(u,S)}/e^{h_0(u,S)}$. From the estimates in Table 5, $\lambda_S = 1 - e^{-0.5403}/e^{-1.6458} = -2.0207$ in Regime 1 and $\lambda_S = 1 - e^{-0.3500}/e^{-1.2675} = -1.5030$ in Regime 2. With the estimate of the market price of regime-switching risk, the risk premium associated with regime-switching shocks can be obtained by the second term in (20). Figure 6 plots the estimated regime-switching risk premiums during the sample period. These

Table 6 Expectation-hypothesis regression using simulated interest rates [a]

	i+j=0.5	i+j=1	i+j=2	i+j=3	i+j=4	i+j=5
j=0.25	-1.9169					
	(0.1323)					
j=0.5		-1.7966				
		(0.1033)				
j=1			-1.6602	-1.8859	-2.1152	-2.3428
			(0.0836)	(0.0868)	(0.0909)	(0.0958)
j=2				-1.2804	-1.4769	-1.6721
				(0.0680)	(0.0696)	(0.0717)
j=3					-1.0817	-1.2511
					(0.0625)	(0.0.0636)
j=4						-0.9612
						(0.0600)

[a] This table reports the estimate of β_{ij} in regression $i(R_{t+j}^i - R_t^{i+j}) = \alpha + \beta_{ij}[j(R_t^{i+j} - R_t^j)] + \varepsilon_{t+j}^i$ where R_t^k is the k-year interest rate at time t *using the simulated interest rates* according to the estimated term structure model in Section 2.5. Under the null hypothesis of the expectation theory, $\beta_{ij} = 1$. Numbers in parentheses are Newy-White standard errors.

Table 7 The estimated bond risk premiums and the business cycle

	RPY1	RPY2	RPY3	RPY4	RPY5	BC
RPY1	1					
RPY2	0.9776	1				
RPY3	0.9446	0.9925	1			
RPY4	0.9159	0.9799	0.9969	1		
RPY5	0.8947	0.9687	0.9917	0.9987	1	
BC	-0.1290	-0.1680	-0.1872	-0.1980	-0.2043	1
$\hat{\alpha}_i$	0.1663**	0.3978**	0.6884*	1.0270*	1.4032	
	(0.0799)	(0.1999)	(0.3630)	(0.5638)	(0.7937)	
$\hat{\beta}_i$	−0.0028	−0.0100**	−0.0210**	−0.0352**	−0.0517**	
	(0.0017)	(0.0043)	(0.0078)	(0.0120)	(0.0169)	
$\hat{\gamma}_i$	-0.0550	-0.1059	-0.1544	-0.1996	-0.2441	
	(0.0883)	(0.2210)	(0.4313)	(0.6232)	(0.8774)	
Adjusted R^2	0.038	0.051	0.057	0.060	0.062	

[a] The first 6 rows of this table report the correlation coefficients among the estimated bond risk premiums and the business cycle. RPY1, RPY2, ..., RPY5 are the instantaneous expected excess return on 1-year, 2-year, ..., 5-year bonds, respectively, given in (20) in Section 2.6. BC is the dummy variable for the business cycle with BC=1 indicating an expansion and BC=0 indicating a recession. The last 4 rows report the OLS regression $\text{RPY}i_t = c + \alpha_i\text{SP}_t + \beta_i\text{BC}_t + \gamma_i[\text{SP}_t \times \text{BC}_t] + \varepsilon_{i,t}$, where RPY$i$ is the risk premium on the i-year bond, SP is the yield spread between the 5-year bond and the 1-month Bill. Numbers in parentheses are Newy-West standard errors. An $*$ indicates the estimate is significant at 10% level. An $**$ indicates the estimate is significant at 5% level.

values seem economically important and suggest that the regime-switching risk is indeed priced by bond investors. The standard errors of the estimate of ϕ in both regimes, however, are very big (5.0976 and 2.3850 respectively). In fact, among all the estimated parameters, ϕ is the least accurately estimated one. Notice that $\lambda_S = 0$ when $\phi(u,S) = h_0(u,S)$. With the larger standard errors, we are in fact not able to reject that $\phi(u,S) = h_0(u,S)$, or $\lambda_S = 0$ in both regimes. The uncertainty regarding the regime-switching risk premiums may have reflected the caveat of affine models under regime shifts as we discussed above. In order to obtain a closed-form solution of the term structure of interest rates in an affine model, we have to restrict the risk-neutral regime-switching probabilities to be constant. This implies that the model would force the regime-switching risk to be priced if the regime-switching probabilities are allowed to be time-varying under the physical probability measure. In this paper we choose not to impose a non-zero market price of regime-switching risk a prior by using a less general specification of the regime-switching probabilities. Our empirical results suggest that more studies are needed in order to get a better assessment of the regime-switching risk premiums.
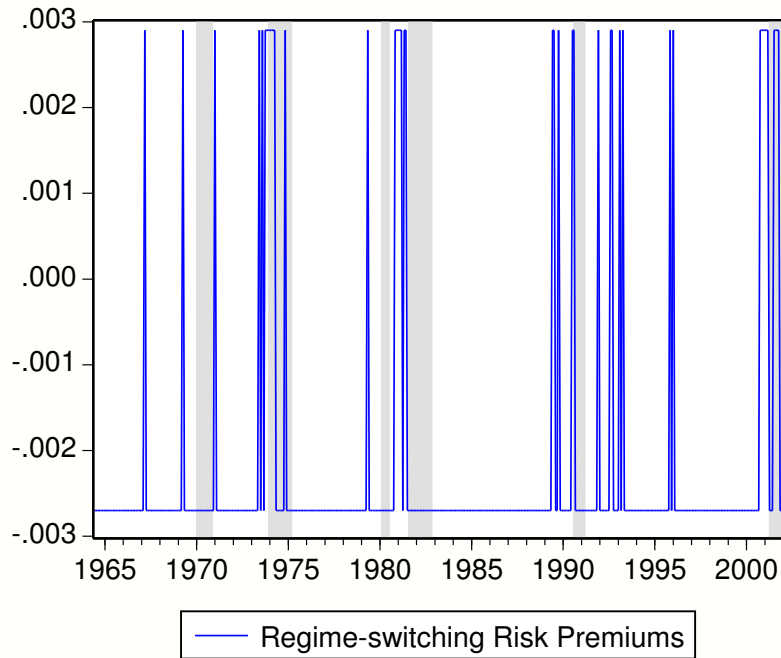

Fig. 6 Regime-switching risk premiums. The figure plots the estimated regime-switching risk premiums on the 5-year bond during 1964-2001. The shaded areas are economic recessions dated by NBER.

4 Conclusion

Using an analytically simple representation of Markov regime shifts, this paper develops and estimates a continuous-time affine model of the term structure of interest rates under the risk of regime-switching. The model elucidates the dynamic effects of regime shifts on the yield curve and bond risk premiums. The empirical results show that the model is able to account for many salient features of the term structure of interest rates and confirm that regime-switching indeed seems to be an important property of interest rate movements. There are still some uncertainties regarding the magnitude of the regime-switching risk premiums that warrant the development of more general dynamic models of the term structure of interest rates under regime shifts.

In the current model, regimes, though a latent variable to econometricians, are assumed to be observable to bond investors. A natural extension is to assume that the regimes are not observable to bond investors either, and that the investors must learn the regimes through other observable state variables. One example is that the regimes may represent different stances of the monetary policy and bond investors must infer from different signals about the true intentions of the central bank.

It should be noted that the model developed in the current paper is an empirical one. The regimes identified by the model lack clear structural interpretations. Another extension of the present paper is to incorporate the model of the term structure of interest rates into a well specified macroeconomic model with regime shifts. With such a structural model, we will be able to identify and interpret different regimes in terms of macroeconomic fundamentals. These extensions are left for future studies.

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