Homework 1, ORFE 569

Due Feb. 27, 2007

1. Suppose that X_t is a compound Poisson process, namely, $X_t = \sum_{k=1}^{N(t)} \xi_k$, where $N(t)$ is a Poisson process with intensity λ and ξ_k are i.i.d. random variables with density $p(y)$.
   Show its (infinitesimal) generator is
   $$\mathbf{A}f(x) = \lambda \int \big[f(x+y) - f(x)\big]p(y)dy.$$
   for $f \in \mathcal{D}_A$ (bounded f).
2. For biasing rule (I), calculate the rest of $p(y|x)$ and present the whole $p(y|x)$.
3. For biasing rule (II), calculate $p(y|x)$ and present the whole $p(y|x)$.
4. Explicitly solve the general linear SDE (LSDE):
   $$dX_t = (A_t + B_t X_t)dt + (C_t + D_t X_t)dW_t$$
   where W_t is a standard Brownian motion, and A_t, B_t, C_t and D_t are deterministic functions. Let
   $$K_t = \int_0^t (B_s - \frac{1}{2}D_s^2)ds + \int_0^t D_s dW_s$$
   Precisely, show
   $$X_t = e^{K_t}\Big(X_0 + \int_0^t e^{-K_s}(A_s - C_s D_s)ds + \int_0^t e^{-K_s} C_s dW_s\Big).$$
   (**Hint:** Apply two-dimensional Itô formula for Itô diffusions to $d(X_t e^{-K_t})$).
5. Let X_t be a two dimension Itô diffusion
   $$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$$
   where $X_t, \mu(X_t)$, and B_t are all 2×1 vectors and $\sigma(X_t)$ is a 2×2 matrix. Show its (infinitesimal) generator is
   $$\mathbf{A}f(x) = \sum_{k=1}^{2} \mu_k(x)\frac{\partial f}{\partial x_k} + \frac{1}{2}\sum_{1\le i,j\le 2}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x}^T \mu(x) + \frac{1}{2}\mathrm{tr}\big[\sigma(x)\sigma^T(x)\frac{\partial^2 f}{\partial x^2}\big],$$
   for bounded and twice continuously differentiable f.
6. Heston Model (RFS 1993):
   $$\begin{cases} \frac{dX_t}{X_t} = \mu dt + \sigma_t dW_t \\ dV_t = (\gamma - \alpha V_t)dt + \kappa\sigma_t dB_t \end{cases}$$
   where $V_t = \sigma_t^2$, W_t and B_t are correlated standard Brownian motion with correlation coefficient ρ. Determine its generator $\mathbf{A}f(v,x)$.
7. The limiting diffusion model of AR(1) Exponential ARCH (Nelson 1990, JEM):
   $$\begin{cases} d[\ln(X_t)] = c\sigma_t^2 dt + \sigma_t dW_t \\ d[\ln(V_t)] = (\gamma - \alpha\ln(V_t))dt + \kappa dB_t \end{cases}$$
   where $V_t = \sigma_t^2$, W_t and B_t are correlated standard Brownian motion with correlation coefficient ρ. Determine its generator $\mathbf{A}f(v,x)$.