Homework 3, ORFE 569
Due Mar. 29, 2007

1. Let X be a semimartingale, $X_0 = 0$. Suppose that Z satisfies the equation $Z_t = 1 + \int_0^t Z_{s-} dX_s$. Show that

$$Z_t = \exp\left\{X_t - \frac{1}{2}[X,X]_t\right\} \prod_{0<s\le t} (1+\Delta X_s) \exp\left\{-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right\}.$$

(Hint: Let $K_t = X_t - \frac{1}{2}[X,X]_t^c$ and $S_t = \prod_{0<s\le t}(1+\Delta X_s)\exp\{-\Delta X_s\}$. First show that $Z_t = S_t e^{K_t}$. Then, apply Itô lemma to $f(K_t, S_t)$ where $f(x,y) = ye^x$ to show Z_t satisfies the SDE.)

2. Show the SDE

$$L(t) = 1 + \sum_{k=1}^n \int_0^t \Big[\lambda_k(\theta(s-), X(s-), s-) - 1\Big] L(s-) d(Y_k(s) - s),$$

has the solution

$$L(t) = \prod_{k=1}^n \exp\left\{\int_0^t \log \lambda_k(\theta(s-), X(s-), s-) dY_k(s) - \int_0^t \Big[\lambda_k(\theta(s), X(s), s) - 1\Big] ds\right\}.$$

using two approaches:

   (a) Use Itô lemma;

   (b) Use the formula in Problem 1.

3. Derive the recursive algorithm to compute the Bayes estimates of your model for Lab Assignments 2 and 3. (Note that an recursive algorithm includes both $p(y_k|x)$ and the recursive equations derived from the filtering equation.)

4. Derive the recursive algorithm to compute the Bayes estimates of another model of your interest among the possible models given.

5. Let N be a Poisson process with intensity λ, and let $\{\xi_k\}$ be a Bernoulli sequence with $P(\xi_k = 1) = 1 - P(\xi_k = 0) = p$. Define

$$N_1(t) = \sum_{k=1}^{N(t)} \xi_k, \qquad N_2(t) = \sum_{k=1}^{N(t)} (1 - \xi_k).$$

Prove that N_1 and N_2 are independent Poisson processes with parameter λp and $\lambda(1-p)$ respectively.

6. Assume $dP/dQ = L$ on $(\Omega, \mathcal{F})$. Prove that

   (a) $E^P[X] = E^Q[XL]$ (Hint: Use Radon-Nikodym theorem)

   (b) (**Extra Credits**) the corresponding formula for conditional expectation: Suppose $\mathcal{D}$ is a sub-σ-field of $\mathcal{F}$. Then,

$$E^P[Z|\mathcal{D}] = \frac{E^Q[ZL|\mathcal{D}]}{E^Q[L|\mathcal{D}]}.$$